# THE BEGINNER'S ALGEBRA

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# THE

# BEGINNER'S ALGEBRA

BY

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# PREFACE

THIS book aims to give a working knowledge of elementary algebra. It supposes in the pupil such knowledge of arithmetic as is usually obtained before entrance to the highest grade in the grammar school. It introduces multiplication immediately after addition, and teaches factoring while teaching multiplication. It insures by this arrangement a saving of time, accuracy in the use of signs, and an early familiarity with the structure of those expressions which form the staple of algebra. It treats the equation as derived from the problem. It leads to the problem by exercises in expression and in forming equalities. It helps the pupil to become familiar with problems of the ordinary types by means of exercises in their construction. Its purpose throughout is to tell the pupil little that he can see, and to help him to see by employing well-directed questions. It is a teaching book, useful to any beginner.

## CONTENTS

## SUGGESTIONS TO TEACHERS

THE BEGINNER'S ALGEBRA indicates the middle course between developing the subject without a text-book and using a text-book that leaves nothing to be discovered. The pupil is given only so much information as is needed in making a beginning; he is then given an example to study. His attention is directed and held by tests. These tests are the pupil's lesson, which he is to prepare if possible without aid. No explanation should be given until the pupil has asked a definite question, and has shown that he has done for himself all that he can do.

A part of each recitation period might be devoted to individual needs, but the pupil should go to the teacher to state his difficulties. The conference will strengthen the impression which the teacher wishes to make that the lesson is for each pupil a personal interest; further, the teacher will avoid answering questions before the class, and thereby encouraging the inert to wait for what will be brought to them.

The class work should be so conducted that there will be the least possible temptation for one pupil to offer the results of another's study. For the recitations new examples and similar or other appropriate questions should follow the tests of the book.

When the exercises set for him have been done, the pupil should write the method. The superfluous or the inadequate in his expressions should be indicated, but he should make the correction. Several of the best of the rules and the axioms which are handed in should be put upon the board, and their excellences in clearness, conciseness, and orderly arrangement should be pointed out.

The exercises of school algebra involve chiefly numbers which are powers or are products easily factored. The beginner should be drilled in these number forms with the care bestowed upon declensions and conjugations in language. They are the current expressions of algebra. They should be recognized at sight, resolved without hesitation, and their possibilities in ordinary combinations should be known as well. This is the work of multiplication; of multiplication not confined to the process of multiplying, but devoted to the structure and relations of the polynomials in most frequent use.

Throughout the chapter on multiplication there should be review with every exercise; every lesson should be cumulative. The pupils who do not easily recognize and use the numbers taught in the past lessons would profit by being appointed to select exercises for the reviews. The indifferent pupil might be moved by having pointed out to him in the exercises of the following fifty pages the frequent recurrence of the number forms which he should learn in Chapter III.

But what is not effected by inspiration must be accomplished by drill. No pupil should be allowed to do with his pencil one exercise after another, and then to drop all interest in each with the waste paper upon which he has done his work.

Numbers and their products, numbers and their powers, should often be repeated aloud. While many pupils recall what they have seen, others more readily recall what they have heard and have repeated.

Quickness of perception in dealing with monomials, such as Exercise 32 requires, can be cultivated before division is reached. The pupils who most need the advantage that may be derived from preparing the exercises should place as many as are needed upon the blackboard for several successive recitations.

Although the pupil is to be encouraged in that visualizing which aids his conception of the problem before him, in solving the problem he should keep in mind that the notation is the expression of numbers, the equation is the equality of numbers, the result is a number, and that all his written expressions should be accurate statements consistent with these facts.

The pupil must be made to acquire correct habits of thought and expression in the beginning. If possible he should be convinced that thought and expression react upon each other, and that confused thought may be the penalty for heedless, obscure, and ill-arranged solutions.

In giving free play to the pupil's powers of observation and reflection, in strengthening his self-reliance, in holding him to accuracy and promptness, the disciplinary value of the study of algebra is conserved for him by both text-book and teacher. But it is by his own effort that he acquires the best and most enduring knowledge and skill.

With every new text-book there is the probability that it will reach some one inexperienced who will be interested to look at the task of teaching the subject as it is seen from another's point of view; hence these suggestions.

CLARIBEL GERRISH.

JUNE, 1902.

# ALGEBRA

## I. Definitions. Notation

**1.** In **Algebra**, numbers are represented by figures, by letters, or by combinations of letters and figures.

**2.** A number represented wholly or in part by letters is called a *literal number;* as, $acn$, $5\,cd$, $2\,d + 3\,c$.

**3.** If $a$, $c$, and $n$ represent the digits of a number, the number is written $100\,a + 10\,c + n$, and $acn$ is the product of its digits.

The number 356 may be written in the same form, $300 + 50 + 6$, but the product of the digits is indicated by the signs of multiplication; as, $3 \times 5 \times 6$ or $3 \cdot 5 \cdot 6$.

**4. Coefficient.** If a number is resolved into two factors, and one factor is used to show how many times the other is taken, it is called the *coefficient;* as, in $2\,amx$, 2 may be the coefficient of $amx$, or $2\,am$ may be the coefficient of $x$. The coefficient 1 is omitted.

**5. Terms.** The parts of a number which are preceded by the signs + or – are called *terms.*

6. A number of one term is a *monomial;* of more than one term, a *polynomial.* A polynomial of two terms is a *binomial;* of three terms, a *trinomial.*

7. **Symbols of Quality.** Besides being signs of operation, the signs + and − are *symbols of quality*, and they indicate opposite states or directions. If gain is plus, loss is minus; if deposit is plus, withdrawal is minus; if one direction is plus, the opposite is minus.

8. A number having the sign + is a *positive number;* one having the sign − is a *negative number.*

The sign + may be omitted at the beginning of a number; the sign − is never omitted.

9. **Exponent.** If a literal number has several factors of the same kind, the factors are collected by the use of a figure or a letter called the *exponent*, which shows how many there are; as, $aaaabbc$ is written $a^4b^2c$, and it is read, $a$ fourth power, $b$ second power, $c$ first power. The exponent 1 is omitted.

$$a^4 = a \times a \times a \times a, \text{ but } 4\,a = a + a + a + a.$$

10. **Similar Numbers.** When numbers have the same literal factors of the same power, the numbers are *similar.* $4\,a^2bc^3$, $5\,a^2bc^3$, and $-a^2bc^3$ are similar monomials, or terms.

11. **Intern.** An enclosed algebraic number is an intern′. The enclosing marks are the curves ( ), the brackets [ ], and the braces { }. The vinculum is used for a similar purpose.

## TEST A

Given $4a^2bc - 3am^4 + dx - 7$

**1.** How many terms has the given number? (§ 5)

**2.** What is the greatest number of literal factors in any given term? Write the literal prime factors of each term. How many? What is the sum of the exponents in each term? (§ 9)

**3.** How many times is $b$ used as a factor? (§ 9)

**4.** Why has $4a^2bc$ no sign? (§ 8)

**5.** Why has $c$ no exponent? (§ 9)

**6.** What kind of number is each of the first three terms? (§ 2)

**7.** Of what kind is the last term considered without its sign? Considered with reference to its sign? (§ 8)

**8.** How many times is $dx$ taken? $am^4$? (§ 4)

**9.** If these four terms represent the results of trade, which indicate loss? (§ 7)

**10.** Make a number of $x$, $y$, and $z$ used as factors.

**11.** Make a number of $x$, $y$, and $z$ used as digits of the number. (§ 3)

**12.** If $a = 5$ and $n = 6$, what is the numerical value of $an$? Of $na$? Of $10a + n$? Of $10n + a$?

**13.** Does a change in the order of factors in a literal number change the value of the number?

**14.** Does a change in the order of digits in a literal number change the number? Suppose, in Ex. 11, that $x = y$, and write two forms of the number.

**15.** Transform $a^3b^2c$, $a^2bc^3$, and $ab^3c^2$ into similar terms by a different use of the exponents 1, 2, 3. How many sets of similar terms can be made? (§ 10)

**16.** Is $abc$ a product? Is it a power? Is $aab$ a power? Is $bbb$ a power? Is $c$ a power? Is $c$ a product?

**17.** Write in order the prime factors of $5^2c^3d$, $2^4a^2$, and $3^3mp$.

**18.** What is the absolute, or unit value, of $-9a$? Of $+9a$? What is the quality of each? (§ 7)

**19.** If a man says that his thermometer stands at $+9°$, and that his day's trade has left him $-\$5$, what does he mean? Express the opposites of these statements.

**20.** How many terms in $4a^2bc \div 3am^4 \times dx \div 7$? (§ 5)

**21.** What is the meaning of the word coefficient? Of the word polynomial, or multinomial? Of vinculum?

NOTE. — If two or more interns are written without intervening sign, their product is indicated: $(a+2)(cd-b)$.

The enclosing of several terms of a number is called *grouping terms*.

If $3a + (cd - b)$ is formed from $3a + cd - b$, $(cd - b)$ is a group intern; but $(cd - b)$, a factor of $(a+2)(cd-b)$, is a number not considered with reference to its terms.

In $(cd-b)^3$, the number is enclosed as the base of a power, and not as a group of terms.

The given illustrations include the positive intern only; the negative intern is considered in §§ 26 and 29.

The name "intern" indicates the condition of the number, and the name of its marks should not be associated with it, unless reference to marks is necessary in order to distinguish several interns, one included in another. The expressions, "enclose the number," "the factor enclosed," etc., need no allusion to the enclosing marks.

# II. Addition

**12.** In the following algebraic series of numbers, the negative numbers, beginning at zero, decrease by one to the left:

$\cdots -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \cdots.$

As the negative numbers increase numerically, they decrease algebraically; $-3$ is greater than $-4$.

### TEST B

1. Name two numbers in the given series, opposite in quality, that added would give 0; two that would give $+2$; two that would give $-3$.

2. How many of the negative units in $-7$ added to the positive units in $+4$ will give 0? How many negative units are there in excess of the positive? What then is the algebraic sum of $-7$ and $+4$?

3. How many dollars in $+\$4$ and $-\$4$? In $+\$4$ and $-\$2$? In $-\$5$ and $+\$2$?

4. How many shillings in $+7s.$ and $+4s.$? In $-7s.$ and $-4s.$? In $+7s.$ and $-4s.$?

5. In $15b+(-11b)+(-2b)$, which signs are signs of operation? Which are signs of quality? What is the sum of the three numbers? What would be their sum if the signs of quality were changed?

6. How many times is $a + b$ taken in

$$8(a + b) - 2(a + b) - (a + b)?$$

7. When are interns dissimilar? Are $(a + b)^2$, $\overline{a + b}$, $2[a + b]$, $\{a + b\}$ dissimilar?

8. Separate $4a^2bn^2$ into three positive terms.

9. Separate $-11x^3y^4z$ into four negative terms.

10. Write four terms, two positive and two negative, whose sum is $-11x^3y^4z$.

11. If a boy playing marbles should, at different times, (1) gain 5 marbles and have 3 given him; (2) lose 2 and give away 4; (3) gain 3 and lose 3; (4) gain 5 and lose 3; (5) gain 1 and lose 4, what numbers with appropriate signs would represent the five results?

12. Add mentally the following numbers in their given order:

(1) $2 - 4 + 1 - 3$ (2) $3 - 1 + 5 - 3$

(3) $2 - 7 - 1 + 4$ (4) $5 - 1 - 2 + 3$

(5) $-3 + 9 - 7 + 1$ (6) $10 - 1 + 1 - 3$

(7) $3 - 4 + 2 - 5 + 2 - 3 + 7 - 5 - 1$

(8) $11 - 9 + 8 - 13 + 12 - 7$

(9) $20 - 17 - 15 + 12 + 19 - 14 - 18 + 13 - 11 + 16$

### EXERCISE 1

Arrange similar terms in columns, and find their sum:

1. $-3x$, $-3y$, $-3z$, $2y$, $z$, $x$, $2z$, $2x$, $y$.

2. $7x^2$, $-2x$, $3x^2$, $9x$, $-3xy$, $4y^2$, $-5y$, $-2y$, $-2x^2$, $5y^2$, $-y^2$, $-y$, $4xy$, $-x$.

3. $4ax$, $6ab$, $-3ax$, $2a^2b$, $5ax$, $-a^2b$, $11a^2b$, $-3ab$, $5ab$, $-9a^2b$, $ax$, $ab$, $3ax$, $-2ab$, $27$.

**Find the sum of the given polynomials. For convenience begin at the left.**

**4.**
$$\begin{array}{r} + x^3 - 3x^2 + 1 \\ -4x^3 + x^2 - 9 \\ +2x^3 - x^2 + 8 \\ -3x^3 + 10x^2 - 1 \\ \hline \end{array}$$

**5.**
$$\begin{array}{rrrrr} 3d & -7c & +5b & +4x & \\ 5d & +2c & -b & & -3y \\ -d & -c & -2b & -x & \\ -3d & +4c & +3b & & +y \\ \hline \end{array}$$

**6.**
$$\begin{array}{rrr} -3b^2 & +a^2 & -c^3 \\ 9b^2 & +3a^2 & +c^2 \\ -7b^2 & +4a^2 & -c \\ b^2 & +a^2 & \\ \hline \end{array}$$

**7.**
$$\begin{array}{rrr} -(a+b)^2 & -7xy & +5(m+n) \\ 4(a+b)^2 & +xy & +15(m+n) \\ -3(a+b)^2 & +11xy & +2(m+n) \\ \hline \end{array}$$

**8.**
$$\begin{array}{rrr} 3a(m+n)^2 & -5c(x-y)^3 & +7n(a-b) \\ -4a(m+n)^2 & +2c(x-y)^3 & -5n(a-b) \\ 2a(m+n)^2 & -3c(x-y)^3 & -n(a-b) \\ \hline \end{array}$$

**9.** Write the method of finding the sum of polynomials, beginning with the arrangement of terms.

**10.** Add $7x - 5y + 4z$, $x + 2y - 11z$, $3x + y + 5z$, and $5x - 4y - z$.

**11.** Add $6m - 13n + 5p$, $8m + n - 9p$, $m - n - p$, and $m + 2n + 5p$.

**12.** Add $3xy + 2xy^2 - x^2y^2 + 3yz$, $yz - y^2z^2 + 2x^2y^2 - xy^2$, and $y^2z^2 - 3xy^2 - 3y^2z^2$.

**13.** Add $3ax + 5bx + 7cx$, $-2bx - 3cx - ax$, $10ax - 11cx - 12bx$, and $-cx + 4ax + 12bx$.

**14.** Add $a - 1$, $-a^2 + a$, $6 - 4a^2$, $6a^2 - 7a$, $3a^2 + 2a - 5$.

**15.** Add $am + bn - cp$, $am + bn + 3cp$, $3cm - cp - bn$, and $-2bn - 2am - 4cp$.

**16.** Add $3(a^2+b^2)+4(x^2-y^2)$, $-6(x^2-y^2)-4(a^2+b^2)$, $-(a^2+b^2)+(x^2-y^2)$, $2(a^2+b^2)-2(x^2-y^2)$, and $(a^2+b^2)+3(x^2-y^2)$.

**17.** Add $(a+b)x-4a(x+y)-by+a(x+y)-3by+2(a+b)x+(a+b)x-2a(x+y)+2by-7(a+b)x-5a(x+y)+9by-(a+b)x+9a(x+y)-3by$. Add mentally. How many times $x$? $(x+y)$? $by$?

**18.** Add $a^4+4a^3b+6a^2b^2+4ab^3+b^4$ and $6a^2b^2+b^4-4ab^3-4a^3b+a^4$. Add mentally. What opposites occur? How many times $a^4$, $b^4$, and $a^2b^2$ in the result?

**19.** Add without copying: $3x^3-5x^2+6x+7$, $2x^2-9x-8$, $-2x^3+3x^2+4x$, $3x^3-x^2-9x$, $-x^3-x^2-x+4$.

**20.** Add without copying: $-2b^3+3ab^2+a^3$, $5b^3+8a^3$, $-ab^2+5a^2b-3a^3$, $ab^2+9a^2b-2a^3$.

**21.** Add $x^5+2x^4-4x^3-6x^2$, $2x^4+4x^3-8x^2-12x$, and $-3x^3-6x^2+12x+18$.

**22.** Add $x^3y^2-2x^2y^3+3xy^4-y^5$, $-3x^4y+6x^3y^2-9x^2y^3+3xy^4$, and $x^5-2x^4y+3x^3y^2-x^2y^3$.

**23.** Add $7a^2-3ab+a$, $3a^2-b^2+3a-b$, $-2a^2+4ab+5b^2-a-2b$, $4a^2+4b^2-2a$, and $-7ab-b^2+9a-5b$.

**24.** Add $5xy^2-10x^2y^3+20x^3y^4$, $-11xy^2+9x^2y^3+5x^4y^5$, $-7x^2y^3-16x^3y^4+3x^4y^5$, and $6xy^2+8x^2y^3-4x^3y^4-8x^4y^5$.

**25.** Add $a^5+a^4-2a^3+a^2$, $-a^4-a^3+2a^2-a$, and $2a^3+2a^2-4a+2$.

## III. Multiplication

**13.** In multiplication the multiplicand is taken as many times as there are units in the multiplier.

If the multiplier is positive, the product has the quality (§ 7) of the multiplicand; if the multiplier is negative, the quality of the product is the opposite of the quality of the multiplicand. Thus:

$$+6 \times (+2) = +12 \qquad +8 \times (-4) = -32$$

$$-7 \times (+3) = -21 \qquad -9 \times (-5) = +45$$

(1) $+a \times (+b) = +ab$ (3) $+a \times (-b) = -ab$

(2) $-a \times (+b) = -ab$ (4) $-a \times (-b) = +ab$

### TEST C

1. In example (2) what is the quality of the multiplicand? Of the product?

2. In example (3) what is the quality of the multiplicand? Of the product?

3. How many times is the multiplicand taken in each of the four literal products?

4. When should the product have the same sign that the multiplicand has?

5. When should the product have a sign the opposite of the sign of the multiplicand?

6. What is indicated by a negative multiplier besides the number of times the multiplicand is to be taken?

7. Introduce the factor $+c$ into each of the four examples. Is there any change in the quality of the products?

(5) $a \times b \times (-c) = -abc$ (7) $a \times (-b) \times (-c) = +abc$

(6) $-a \times b \times (-c) = +abc$ (8) $-a \times (-b) \times (-c) = -abc$

8. What is the sign of the product when there is one negative factor? When there are two negative factors? When there are three negative factors?

9. Introduce the factor $-d$ into example (8). How does the product obtained agree with the product in example (6)?

10. In examples (5) and (8), what is apparently the law in regard to the number of negative factors that will produce a negative product?

11. In examples (6) and (7) and $(-a)(-b)(-c) \times (-d)$, what is apparently the law in regard to the number of negative factors that will produce a positive product?

**14.** The product of two or more numbers is the product of the prime factors of those numbers.

A prime literal factor occurs in the product as many times as it occurs in both multiplicand and multiplier. Thus:

$$aaa \times aa = aaaaa.$$

But $aaa$ may be written $a^3$, $aa$ may be written $a^2$, and $aaaaa$ may be written $a^5$. (§ 9)

Hence, $a^3 \times a^2 = a^5$; that is, the exponent of $a$ in the product is the sum of the exponents of $a$ in the multiplicand and multiplier.

**EXAMPLES**

1. $a^3 \times a = a^{3+1}$ or $a^4$
2. $ab \times a^2 = a^{1+2}b$ or $a^3b$
3. $-5a \times 2a = -10a^2$
4. $(-3b)(-4b^2) = 12b^3$
5. $(-a^3b)(-3b^3) = 3a^3b^4$
6. $a^2bc^3 \times (-b^5c) = -a^2b^6c^4$
7. $(a+x) \times 2 = 2a + 2x$
8. $(c-d) \times -1 = -c + d$
9. $(a^2 + b^3) \times ab = a^3b + ab^4$
10. $(c - x^2) \times c^2x = c^3x - c^2x^3$
11. $ac(ab - cd) = a^2bc - ac^2d$
12. $-x^2(x - yz) = -x^3 + x^2yz$
13. $-2a(b^2 - a) = -2ab^2 + 2a^2$
14. $3(m-n)(-n) = -3mn + 3n^2$
15. $-x(x-y) \times 2y = -2x^2y + 2xy^2$

**EXERCISE 2**

Multiply:

1. $+a$ by $a^5$.
2. $+b^2$ by $b^3$.
3. $+c^3$ by $c^4$.
4. $-a^3$ by $a^3$.
5. $-a^2$ by $-a^4$.
6. $+a$ by $-a^5$.
7. $+ab^3$ by $-a^4b$.
8. $-ac^3$ by $-3ab$.
9. $+2ab$ by $a^2b^3$.
10. $-5ab$ by $-3ab$.
11. $+4x^2y$ by $7axy$.
12. $-7a^2b$ by $2ab^3$.
13. $+8xy^2$ by $-3xyz$.
14. $-7x^2y^2$ by $-x^3y^4$.
15. $-a^2x$ by $11a^3x^5$.
16. $-3a^2b^2c^2$ by $-bc$.
17. $-abx$ by $-aby$.
18. $-5ay^2$ by $2a^2x^3$.
19. $8ab^3(-3a^2b)(-3ab) = ?$
20. $-7ab^2(-5a^2b)(-abc) = ?$

**TEST D**

1. In multiplying one monomial by another, what law of signs is observed? In what order are letters written? In what way is shown the number of times a letter occurs as a factor in both multiplicand and multiplier?

2. If $-a$ is used as a factor 3, 5, 7, or 9 times, what sign will the resulting power have? If $-a$ is used 2, 4, 6, or 8 times?

3. Give three forms of multiplicand and multiplier that will produce $+b^6$.

4. Give three groups of three factors each that will produce $-a^5b^4$.

5. Resolve $-48$ into its prime factors, and state how many of them may have been negative, and how many must have been negative, to produce the number.

6. In the product $+a^8$, how many negative factors are possible?

7. What factors of 36 added give $-15$? What factors of $-9$ added give 0? Of $+9$ give $-6$? Of $+9$ give $-10$? Of $+9$ give $+10$?

8. What negative factor was repeated to produce the following powers: $a^8$, $-m^3$, $-27$, $c^4$, $b^2$, $-x^7$, $-(acd)^3$, $-(a+b)^5$, $(a+b)^2$?

9. What is true of the exponents of the above positive powers, and of the exponents of the negative powers? If all term signs were changed, would question 8 be appropriate? Are $-a^8$, $-c^4$, $-b^2$ powers? Are $\pm m^3$, $\pm 27$, $\pm x^7$ powers?

10. Does $2^4 \times 3^4 = (2 \times 3)^4$? Does $(5 \times 6)^2 = 5^2 \times 6^2$? Does $(a \cdot c \cdot d)^3 = a^3 \cdot c^3 \cdot d^3$?

11. Does $2^n \times 3^n = (2 \times 3)^n$? Does $(a \cdot c \cdot d)^m = a^m \cdot c^m \cdot d^m$?

12. If $a^3 \times a^2 = a^{3+2}$, does $a^m \times a^n = a^{m+n}$? Does $a^3 \times a^n = a^{3+n}$?

13. By what was $a^3$ multiplied to make $a^5$? By what was $a^{3m}$ multiplied to make $a^{5m}$?

14. By what was $p + q$ multiplied to make $p^2 + pq$?

15. By what was $a - bc$ multiplied to make $-a^3 + a^2bc$?

16. By what was $a^m + b^m$ multiplied to make $a^{m+2} + a^2b^m$?

17. Does $-3$ equal $-1 \times 3$? Does $-3$ equal $(-1) + (-1) + (-1)$? What does $(-1)^3$ equal?

18. If $m = 1$, $n = 2$, and $a = 3$, what does $a^{m+n}$ equal? What does $a^{2m} \times a^{2n}$ equal? What does $a^m \times a^{2n} \times a^{m+1}$ equal?

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**15.**

(a)

$$\begin{array}{l} a + b \\ \underline{a + b} \\ a^2 + ab \\ \quad\underline{+ ab + b^2} \\ a^2 + 2\,ab + b^2 \end{array}$$

(b)

$$\begin{array}{l} a - b \\ \underline{a - b} \\ a^2 - ab \\ \quad\underline{- ab + b^2} \\ a^2 - 2\,ab + b^2 \end{array}$$

1. In examples (*a*) and (*b*) what is the object in the arrangement of the partial products?

2. What is a polynomial of two terms? Of three terms?

3. If $a - b$ is multiplied by $a - b$, what name other than product is given to the result?

4. What is meant by the expression "trinomial square"?

5. If a monomial is multiplied by a monomial, how many terms are there in the product? What number multiplied by the same number will produce a binomial? Is $4x^2y^2$ a second power? Is $a^2+b^2$? Is $a^2+2ab+b^2$?

6. Is $p^2+2pq+q^2$ the square of $p+q$? (Study examples (*a*) and (*b*), and learn how each term of a trinomial square is produced.)

**EXERCISE 3**

Find the indicated products and second powers:

1. $(b-d)(b-d)$
2. $(m+3)(m+3)$
3. $(c+5)(c+5)$
4. $(1-a)(1-a)$
5. $(x+2)(x+2)$
6. $(2x+y)(2x+y)$
7. $(5-3)(5-3)$
8. $(3a+2b)(3a+2b)$
9. $(n-a)(n-a)$
10. $(s+3)(s+3)$
11. $(k+0)(k+0)$
12. $(z-x)(z-x)$
13. $(am+3)(am+3)$
14. $(ab+cd)(ab+cd)$
15. $(x-y)^2$
16. $(m+n)^2$
17. $(c-z)^2$
18. $(3-x)^2$
19. $(a+d)^2$
20. $(10a+n)^2$
21. $(10n+a)^2$

**EXERCISE 4**

Enclose each of the binomial factors that produce the following trinomials:

1. $m^2+2mx+x^2$
2. $c^2+2cx+x^2$
3. $a^2-4a+4$
4. $a^2+6a+9$
5. $a^2b^2-2ab+1$
6. $x^2+12x+36$
7. $1-4x+4x^2$
8. $6ab+a^2+9b^2$
9. $4c^2+9a^2+12ac$
10. $25a^2b^2-10abc+c^2$

**16.** *Example.*

$$
\begin{array}{l}
a - b - c \\
a - b - c \\
\hline
a^2 - ab - ac \\
\quad - ab + b^2 + bc \\
\quad\quad - ac + bc + c^2 \\
\hline
a^2 - 2\,ab - 2\,ac + b^2 + 2\,bc + c^2
\end{array}
$$

or, $a^2 + b^2 + c^2 - 2\,ab - 2\,ac + 2\,bc.$

**EXERCISE 5**

Multiply, and rearrange the terms of the product:

1. $(x - y + z)(x - y + z)$
2. $(a - d - c)(a - d - c)$
3. $(m + n + c)(m + n + c)$
4. $(b + l + k)(b + l + k)$
5. $(p + q - 4)(p + q - 4)$
6. $(5 - t - v)(5 - t - v)$

**TEST E**

1. How many terms in the product of
$$(a - b - c) \times (a - b - c)?$$
2. How many terms are second powers?
3. Why are the second powers positive?
4. How are the other terms formed?
5. Does changing the order of the terms change the value of the product?
6. Give a rule for writing the square of a trinomial.

**EXERCISE 6**

Write the indicated second powers:

1. $(n - a + b)^2$
2. $(p + o + m)^2$
3. $(m - a - x)^2$
4. $(a - b + 2)^2$
5. $(c + d - 3)^2$
6. $(x^2 + x + 1)^2$
7. $(1 - a - a^2)^2$
8. $(x^3 + x^2 + x)^2$

**17.**

| (a) | (b) | (c) |
|---|---|---|
| $a+b$ | $x+2$ | $3+n$ |
| $a-b$ | $x-2$ | $3-n$ |
| $a^2+ab$ | $x^2+2x$ | $9+3n$ |
| $\quad -ab-b^2$ | $\quad -2x-4$ | $\quad -3n-n^2$ |
| $a^2-b^2$ | $x^2-4$ | $9-n^2$ |

1. What name not common to the three numbers can be given to $a+b$? To $a-b$? To $a^2-b^2$?

2. How do $a-b$ and $a^2-b^2$ differ?

3. Contrast example (*a*), § 17, with example (*a*), § 15. Why is $b^2$ positive in one result and negative in the other?

### EXERCISE 7

1. $(m+n)(m-n)=?$
2. $(3+c)(3-c)=?$
3. $(a+d)(a-d)=?$
4. $(xy+d)(xy-d)=?$
5. $(3x+my)(3x-my)=?$
6. $(px^2+3)(px^2-3)=?$
7. $(bx+cy)(bx-cy)=?$
8. $(y+cz)(y-cz)=?$
9. $(nx+4y)(nx-4y)=?$
10. $(xy+1)(xy-1)=?$

### EXERCISE 8

Give the product of the sum and the difference of the following numbers:

1. $a$ and $h$
2. $ac$ and $bd$
3. $m$ and $z$
4. 11 and $c$
5. $m$ and $b$
6. $y$ and 5
7. $x^3$ and 1
8. $m^2$ and 3
9. $abc$ and 1
10. 2 and $pxy$
11. $(a-1)$ and $(b+2)$
12. $(x+q)$ and $(n-z)$
13. $(1-yz)$ and $(ts+1)$
14. $(p-r)$ and $(w+2)$
15. $(A+B)$ and $(C+D)$

## THE CUBE OF A BINOMIAL

**18.** (*a*) Expand $(a+b)^3$, or multiply

$$(a+b)\times(a+b)\times(a+b).$$

(*b*) Expand $(a-b)^3$, or multiply

$$(a-b)\times(a-b)\times(a-b).$$

(*c*) Expand $(c+2)^3$, or multiply

$$(c+2)\times(c+2)\times(c+2).$$

1. What are the equal factors of $x^3+3x^2y+3xy^2+y^3$?

2. What power of $x-y$ is $x^3-3x^2y+3xy^2-y^3$?

3. The third power of what binomial is

$$n^3-3n^2\cdot 5+3n\cdot 5^2-5^3?$$

## TEST F

1. How many terms in the third power of a binomial?

2. How many units in the index, or exponent, of this power? How does the number of terms compare with the number of units?

3. What is the exponent of $x$ in the first term of the given powers? Of $y$ in the last term?

4. What is the numerical coefficient in the second term? In the last but one?

5. What are the term signs in the power of $(x+y)$? Of $(x-y)$?

6. What are the exponents of $x$ and of $y$ in the terms considered from left to right?

### EXERCISE 9

Write the indicated power without multiplying:

| | | |
|---|---|---|
| 1. $(m+n)^3$ | 5. $(b-l)^3$ | 9. $(x-3)^3$ |
| 2. $(c-a)^3$ | 6. $(p+q)^3$ | 10. $(x-2)^3$ |
| 3. $(x-t)^3$ | 7. $(1-x)^3$ | 11. $(2-y)^3$ |
| 4. $(d+n)^3$ | 8. $(y-1)^3$ | 12. $(a+5)^3$ |

---

## 19. POWERS OF BINOMIALS

(a) $(m-n)^4 = m^4 - 4\,m^3n + 6\,m^2n^2 - 4\,mn^3 + n^4.$

(b) $(x-y)^5 = x^5 - 5\,x^4y + 10\,x^3y^2 - 10\,x^2y^3 + 5\,xy^4 - y^5.$

(c) $(a+b)^6 = a^6 + 6\,a^5b + 15\,a^4b^2 + 20\,a^3b^3 + 15\,a^2b^4 + 6\,ab^5 + b^6.$

### TEST G

1. How does the sum of the exponents in any term compare with the index, examples (a), (b), (c)?

2. If the exponent of one letter in a term is known, how can the exponent of the other letter be found?

3. What is the sum of the exponents in any term of the 7th power?

4. In the 3d and the 5th power, how do the coefficients of the first half and the last half of the terms compare? What difference in this respect in the coefficients of the 4th and of the 6th power?

5. In example (b) what must be added to the exponent of $y$ in any term to give the number of the term, counting from the left?

6. What is the number of the term $+21\,x^5y^2$?

7. Can one tell if this term belongs to a power of $x+y$ or to a power of $x-y$? Could one tell from the term which follows $+21\,x^5y^2$?

8. What is the sum of the exponents in $+21\,a^5y^2$? To what power of the binomial does $+21\,a^5y^2$ belong? What is the number of the term? What sign have the odd-number terms in the 3d, 4th, 5th, and 6th powers, whether the power of a sum, as $a+b$, or of a difference, as $x-y$?

9. Given, index 8. How many terms in the power? What is the exponent in the first and in the last term? What is the sum of the exponents in any term? What is the coefficient of the 2d and of the 8th term?

10. What is true of the 4th term of every power of $x-y$ from the 3d power to the 100th power?

11. In the given 4th, 5th, and 6th powers multiply the coefficient of the second term by the exponent of $m$, of $x$, or of $a$, in that term, and divide by the number of the term. With what do the quotients agree? Do the same in the third term.

12. If the exponent of $p$ in a term of the 11th power of $p+q$ is 9, what is the exponent of $q$ in that term? If the coefficient of that term is 55, what is the coefficient of the following term?

13. If one knows the second coefficient from the beginning, does he know the second from the end? Is the coefficient the third from the beginning the same as the third from the end? In writing powers of $p+q$, how many different coefficients must be found for the 7th power? For the 8th power?

14. Write *rapidly* the powers indicated: $(a-b)^2$, $(x-c)^3$, $(a-y)^4$, $(m-n)^5$, $(c-y)^6$, $(a-d)^7$.

20. (*a*)
$$\begin{array}{l} a+4 \\ \underline{a-3} \\ a^2+4a \\ \underline{\quad -3a-12} \\ a^2+a-12 \end{array}$$

(*b*)
$$\begin{array}{l} x-7 \\ \underline{x+2} \\ x^2-7x \\ \underline{\quad +2x-14} \\ x^2-5x-14 \end{array}$$

(*c*)
$$\begin{array}{l} b+12 \\ \underline{b+\ \ 4} \\ b^2+12b \\ \underline{\quad +\ \ 4b+48} \\ b^2+16b+48 \end{array}$$

(*d*)
$$\begin{array}{l} n-6 \\ \underline{n-5} \\ n^2-6n \\ \underline{\quad -\ \ 5n+30} \\ n^2-11n+30 \end{array}$$

### TEST H

1. How are the binomials in each of the given examples alike? How do they differ?

2. What is the sum of the second terms? Their product?

3. Where do the sum and the product of these binomial terms appear in the trinomial product?

4. What factors of $-12$ added give $+1$? Of $-14$ added give $-5$? Of $+48$ added give $+16$? Of $+30$ added give $-11$?

5. If a positive and a negative number are added, and the sum is negative, which of the numbers added has the greater unit value?

6. What is indicated by the sign of the third term in examples (*c*) and (*d*) when considered in connection with the sign of the second term?

### EXERCISE 10

Write the products of the given binomials, and then multiply the binomials to test the results:

1. $(m-4)(m-2)$
2. $(c+8)(c-3)$
3. $(b-7)(b+2)$
4. $(x-9)(x+3)$
5. $(a+5)(a-2)$
6. $(a-4)(a-5)$
7. $(d-2)(d-1)$
8. $(bc-5)(bc+1)$
9. $(y-10)(y+3)$
10. $(b-5)(b-3)$
11. $(l-11)(l+1)$
12. $(z+2)(z+1)$
13. $(mn+3)(mn-1)$
14. $(z^2-2)(z^2-3)$
15. $(p-4)(p+7)$

### EXERCISE 11

Find the two unlike binomial factors of the following:

1. $a^2+11\,a+18$
2. $a^2+9\,a+18$
3. $a^2+19\,a+18$
4. $a^2-11\,a+18$
5. $a^2-9\,a+18$
6. $a^2-19\,a+18$
7. $a^2+7\,a-18$
8. $a^2-7\,a-18$
9. $a^2+3\,a-18$
10. $a^2-3\,a-18$
11. $a^2+17\,a-18$
12. $a^2-17\,a-18$

### EXERCISE 12

1. $(ab+4\,c)(ab-3\,c)=?$
2. $(da-5\,c)(da+2\,c)=?$
3. $(m+3\,n)(m-2\,n)=?$
4. $(y+11\,m)(y+m)=?$
5. $(c^2+7\,a)(c^2-6\,a)=?$
6. $(x^3+5\,y)(x^3+2\,y)=?$

## EXERCISE 13

Factor each of the following numbers:

1. $x^2 - 29xy + 54y^2$
2. $a^2 - 20abx + 75b^2x^2$
3. $x^2 - 20xy + 96y^2$
4. $a^2 - 26ab + 169b^2$
5. $a^2 - 12abx - 28b^2x^2$
6. $a^2 - ay - 2y^2$
7. $1 - 9x + 8x^2$
8. $y^6 - 10y^3 + 9$
9. $x^4 - 13x^2 + 40$
10. $1 - 35a + 300a^2$

## EXERCISE 14. REVIEW

From inspection give the indicated products:

1. $(a+2)(a-3)$
2. $(a+2)(a+2)$
3. $(a-3)(a-3)$
4. $(a+2)(a-2)$
5. $(x+y)(x-y)$
6. $(4-x)(4+x)$
7. $(am-5)(am-1)$
8. $(a-4)(a+11)$
9. $(a+b)(a+b)^2$
10. $(a-b)^2(a-b)$
11. $(acd+x)(acd-x)$
12. $(3x^2+y)(3x^2+y)$
13. $(a^3+b^3)(a^3+b^3)$
14. $(9a^2+b^2)(9a^2-b^2)$
15. $(2a^3-3b^3)(2a^3-3b^3)$
16. $(m-n)^3$
17. $(a-1)^3$
18. $(1+x)^3$
19. $(y+1)^3$
20. $(1-y)^3$
21. $(x-y-z)^2$
22. $(ax+ay+az)^2$
23. $(ab+cd)^2$
24. $(x^2+y^2+z^2)^2$
25. $(x^2+y^2)^2(x^2+y^2)$
26. $(y-27)(y-2)$
27. $(x-15)(x-5)$
28. $(y-12)(y-8)$
29. $(b-13)(b-13)$
30. $(x+2)(x-14)$
31. $(x-2y)(x+y)$
32. $(1-7y)(1-y)$

33. $(a^3-9)(a^3-1)$

34. $(n^2-8)(n^2-5)$

35. $(a^4+b^4)(a^4+b^4)$

36. $(a^8+b^8)(a^8+b^8)$

37. $(a^6+b^6)^2$

38. $(x^4+y^4)(x^4-y^4)$

39. $(n^8+x^8)(n^8-x^8)$

40. $(a^6+b^6)(a^6-b^6)$

41. $(a+b)^2(a+b)^2$

42. $(x-y)^2(x-y)^2$

43. $(x+a)^2(x+a)^3$

44. $(a+b)^3(a+b)^2$

45. $(n-a)^3(n-a)^3$

46. $(b-3)^2(b-3)$

47. $(2+y^2)(2+y^2)^2$

48. $(a^2-b^3-c)^2$

49. $(x-2y-3z)^2$

50. $(ab^2c-a^2bc^3)^2$

51. $(1+x^2yz)(1+x^2yz)$

52. $(p^2+\frac{1}{2})(p^2-\frac{1}{2})$

53. $(m+\frac{3}{4})(m+\frac{3}{4})$

54. $(xy-\frac{2}{8})(xy-\frac{2}{8})$

---

21. (a)
$$\begin{array}{l} x^2-xy+y^2 \\ x+y \\ \hline x^3-x^2y+xy^2 \\ \quad +x^2y-xy^2+y^3 \\ \hline x^3+y^3 \end{array}$$

(b)
$$\begin{array}{l} x^2+xy+y^2 \\ x-y \\ \hline x^3+x^2y+xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3-y^3 \end{array}$$

1. A change of how many signs in example (a) gives example (b)?

2. What change in the product follows?

## EXERCISE 15

Multiply the given numbers:

1. $m^2+mn+n^2$ by $m-n$

2. $m^2-mn+n^2$ by $m+n$

3. $c^2+c+1$ by $c-1$

4. $c^2-c+1$ by $c+1$

5. $3^2+3x+x^2$ by $3-x$

6. $3^2-3x+x^2$ by $3+x$

7. $x^2-2x+2^2$ by $x+2$

8. $x^2+2x+2^2$ by $x-2$

9. $x^4+x^2y^2+y^4$ by $x^2-y^2$

10. $x^4-x^2y^2+y^4$ by $x^2+y^2$

## EXERCISE 16

Give a binomial and a trinomial factor of each of the following binomials:

| | | |
|---|---|---|
| 1. $a^3+b^3$ | 11. $a^3b^6+8$ | 21. $(a^1)^3+(n^1)^3$ |
| 2. $c^3-2^3$ | 12. $125-b^3$ | 22. $(a^2)^3+(n^2)^3$ |
| 3. $3^3+x^3$ | 13. $b^3+c^3$ | 23. $(a^3)^3-(n^3)^3$ |
| 4. $m^3-d^3$ | 14. $8\,a^9-27$ | 24. $(b^4)^3-(c^4)^3$ |
| 5. $x^3+4^3$ | 15. $y^3-8$ | 25. $(x^5)^3-(n^5)^3$ |
| 6. $64-a^3$ | 16. $x^9+64$ | 26. $b^6+m^6$ |
| 7. $27+m^3$ | 17. $b^6+c^3$ | 27. $d^6-a^6$ |
| 8. $a^3-125$ | 18. $x^3-512$ | 28. $n^{12}+x^{12}$ |
| 9. $1+y^3$ | 19. $8\,x^3-729$ | 29. $a^{12}-c^{12}$ |
| 10. $b^3-216$ | 20. $27\,a^3+343$ | 30. $x^{24}+b^{24}$ |

**22.**

## EXERCISE 17

(a)
$$\begin{array}{l} x^2-xy+y^2 \\ \underline{x^2+xy+y^2} \\ x^4-x^3y+x^2y^2 \\ \quad +x^3y-x^2y^2+xy^3 \\ \qquad \underline{+x^2y^2-xy^3+y^4} \\ x^4+x^2y^2+y^4 \end{array}$$

(b)
$$\begin{array}{l} (x^2+y^2)+xy \\ \underline{(x^2+y^2)-xy} \\ (x^2+y^2)^2-x^2y^2 \\ x^4+2\,x^2y^2+y^4-x^2y^2 \\ x^4+x^2y^2+y^4 \end{array}$$

Give the product of the trinomial factor of the sum and of the difference of third powers:

1. $(m^2-mn+n^2)\,(m^2+mn+n^2)$
2. $(c^2-c+1)\,(c^2+c+1)$
3. $(a^2-ab+b^2)\,(a^2+ab+b^2)$

4. $(b^2 - bc + c^2)(b^2 + bc + c^2)$

5. $(n^2 - an + a^2)(n^2 + an + a^2)$

6. $(x^2 - 2x + 4)(x^2 + 2x + 4)$

7. $(3^2 - 3x + x^2)(3^2 + 3x + x^2)$

8. $(x^4 - x^2y^2 + y^4)(x^4 + x^2y^2 + y^4)$

9. $(a^4 - a^2b^2 + b^4)(a^4 + a^2b^2 + b^4)$

10. $(m^8 - m^4c^4 + c^8)(m^8 + m^4c^4 + c^8)$

## EXERCISE 18

Give two trinomial factors of the following trinomials:

1. $d^4 + d^2c^2 + c^4$

2. $s^4 + s^2t^2 + t^4$

3. $w^4 + w^2x^2 + x^4$

4. $v^8 + v^4z^4 + z^8$

5. $p^8 + p^4q^4 + q^8$

6. $c^4 + 2^2c^2 + 2^4$

7. $3^4 + 3^2x^2 + x^4$

8. $4^4 + 4^2a^2 + a^4$

9. A change of what sign would (so far as has been observed) make the trinomials of this exercise prime?

10. Contrast the following trinomials, and state what has been learned in regard to each:

(1) $x^2 + 2xy + y^2$

(2) $x^2 - 2xy + y^2$

(3) $x^2 - 5xy + 6y^2$

(4) $x^2 + xy + y^2$

(5) $x^2 - xy + y^2$

(6) $x^4 + x^2y^2 + y^4$

(7) $x^4 - x^2y^2 + y^4$

11. Contrast the following binomials, also their polynomial factors:

(1) $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

(2) $(a \pm b)^3 = (a \pm b)(a^2 \pm 2ab + b^2)$

(3) $a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3)$

(4) $(a - b)^4 = (a - b)(a^3 - 3a^2b + 3ab^2 - b^3)$

(a) $(a^4 + b^4) = ?$

(b) $(a + b)^4 = ?$

(c) $(a^5 + b)^5 = ?$

(d) $(a + b)^5 = ?$

**23.** EXERCISE 19

1. $(a^2+b^2)(a^2-b^2)=?$
2. $(a^4+b^4)(a^4-b^4)=?$
3. $(a^8+b^8)(a^8-b^8)=?$
4. $(a^{16}+b^{16})(a^{16}-b^{16})=?$
5. $(a^6+b^6)(a^6-b^6)=?$
6. $(a^{12}+b^{12})(a^{12}-b^{12})=?$
7. $(a^{18}+b^{18})(a^{18}-b^{18})=?$
8. $(a^{24}+b^{24})(a^{24}-b^{24})=?$

TEST I

1. Give the sum and difference factors of the binomial differences in examples 1, 2, 3, 4.

2. Have the binomial sums in examples 1, 2, 3, 4 been produced by multiplication so far performed? Are they prime?

3. What is true of all the exponents in the eight examples?

4. Resolve the exponents 2, 4, 8, 16 into their prime factors; then find the prime factors of 6, 12, 18, 24. How do the two groups of exponents differ?

5. Can the binomial sums in the first four examples be written in any other form than that of sums of even powers? In how many ways can $a^{16}+b^{16}$ be written?

6. What common odd-number factor have the exponents of the second group? Can the binomial sums in the last four examples be written in any form except that of the sum of even powers?

EXERCISE 20

From inspection give the indicated products:

1. $(a+b)(a^2-ab+b^2)$
2. $(a^2+b^2)(a^4-a^2b^2+b^4)$
3. $(a^4+b^4)(a^8-a^4b^4+b^8)$
4. $(a-b)(a^2+ab+b^2)$
5. $(a^2-b^2)(a^4+a^2b^2+b^4)$
6. $(a^4-b^4)(a^8+a^4b^4+b^8)$

7. $(a^2 - b^2)(a^2 - ab + b^2)(a^2 + ab + b^2)$

8. $(a^4 - b^4)(a^4 - a^2b^2 + b^4)(a^4 + a^2b^2 + b^4)$

9. $(a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2)$

10. $(a^2+b^2)(a^2-b^2)(a^4-a^2b^2+b^4)(a^2+ab+b^2)(a^2-ab+b^2)$

11. $(a^6 + b^6)(a^{12} - a^6b^6 + b^{12})$

12. $(a^6 - b^6)(a^{12} + a^6b^6 + b^{12})$

**24.** COMMON FACTORS

*Example.*

$$\begin{aligned} -35\,a^2b^3c &= -5 \times 7 \cdot a^2 \cdot b^3 \cdot c \\ 49\,a^3b^2n^3 &= 7^2 \cdot a^3 \cdot b^2 \cdot n^3 \\ 21\,ab^5c^2n &= 3 \times 7 \cdot a \cdot b^5 \cdot c^2 \cdot n \\ \hline & \quad 7^1 \cdot a \cdot b^2, \text{ or } 7\,ab^2 \end{aligned}$$

The first given number has six literal factors; hence it is of the sixth degree. The **degree of a term** is found by adding the exponents of the literal factors only. $7^2a^3b^2n^3$ is of the eighth degree.

The **highest common factor** of two or more numbers is the common factor that contains the greatest common numerical factor and the common literal factor of the highest degree.

EXERCISE 21

Find the highest common factor:

1. $11\,x^2y^4z,\ 33\,xy^3z^2,\ 121\,x^2y^3$
2. $10\,a^2b^2,\ 15\,ab^3c^2,\ 25\,a^2bc^3$
3. $42\,m^2y^3,\ 30\,m^2y^4,\ 18\,m^2y^3c$
4. $16\,a^4bc^6d^3,\ 8\,a^3c^5d,\ 12\,a^2b^3c^3d^2$
5. $51\,x^5y^5z^5,\ 69\,xy^4z^4,\ 42\,x^6y^3z^2$
6. $39\,ab^2c^3n,\ 26\,a^4b^3cx,\ 52\,b^5cnx^2$

### EXERCISE 22

Find the common factor or factors:

1. $a^2 - b^2,\ a^3 - b^3,\ a^2 - 2ab + b^2$
2. $a^2 - 4a + 4,\ a^2 + a - 6,\ a^2 - 4$
3. $a^3 + 3a^2b + 3ab^2 + b^3,\ a^3 + b^3,\ a^2 - b^2$
4. $m^4 - n^4,\ m^4 + 2m^2n^2 + n^4$
5. $9 - 6x + x^2,\ 9 - x^2,\ 27 - x^3$
6. $x^2 + xy + y^2,\ x^3 - y^3$
7. $x^6 + y^6,\ x^4 + 2x^2y^2 + y^4$
8. $a^{12} - b^{12},\ a^6 - b^6,\ a^3 - 3a^2b + 3ab^2 - b^3$
9. $a^{16} - b^{16},\ a^8 - b^8,\ a^4 - b^4,\ a^2 - b^2$
10. $c^3 - 8,\ (c-2)^3,\ c^2 - 4c + 4,\ c^6 - 64$
11. $a^2m^2 + 6am + 9,\ a^2m^2 - 9,\ a^2m^2 + 5am + 6$
12. $a^2 + d^2 + c^2 + 2ad - 2ac - 2cd,\ (a+d)^2 - c^2$
13. $x^2 - 2x + 1,\ x^2 - 1,\ x^3 - 3x^2 + 3x - 1$
14. $1 - y^2,\ 1 - 2y + y^2,\ 1 - 3y + 3y^2 - y^3$
15. $7^2 - 42 + 3^2,\ 7^2 - 3^2,\ 7^2 - 77 + 24$
16. $100 - 100 + 25,\ 100 - 25,\ 100 - 80 + 15$
17. $x^4 + y^4,\ x^8 - x^4y^4 + y^8$

### EXERCISE 23

Find the highest common factor:

1. $y^2(x^3 - 9x^2 + 27x - 27),\ y^3(x^2 - 6x + 9)$
2. $a^2b^2c^2 - 1,\ a^2b^2c^2 - 2abc + 1,\ a^2b^2c^2 + 2abc - 3$
3. $15x(x^2 - 9),\ 33x^2(x-3)^2$
4. $a^2 + 4a + 3,\ a^2 + 2a + 1,\ a^2 - 1$

5. $12(x^4 - y^4)$, $18(x + y)^2$, $21(x^2 + 2xy + y^2)$
6. $x^2 - x - 6$, $x^2 + 3x - 18$
7. $a^3(a + 7)$, $a^2(a^2 + a - 42)$, $a^5(a^2 - 49)$
8. $27(a^3 - b^3)$, $9(a^2 + ab + b^2)$
9. $45(a^4 + b^4)$, $36(a^8 + 2a^4b^4 + b^8)$, $18(a^8 - b^8)$
10. $3a^2bc(m^2 - m + 1)$, $18c^4(m^3 + 1)^3$
11. $x^2 + 20x + 36$, $x^2 + 4x + 4$, $x^2 - 4x - 12$
12. $x^2 + 10x + 24$, $x^2 - x - 20$, $x^2 + 8x + 16$
13. $7(x^2 - x + 1)$, $4(x^6 - 1)$, $14(x^3 + 1)$
14. $35a^5c(a + b - c)^2$, $5ay + 5by - 5cy$
15. $x^4 + a^2x^2 + a^4$, $x^3 - a^3$, $4x^2 + 4ax + 4a^2$
16. $16a^8 - b^4$, $8a^6 - b^3$, $2a^3bc - ab^2c$
17. $x^6 + y^6$, $x^4 - x^2y^2 + y^4$, $x^2 - xy + y^2$

### MULTIPLES

**25.** A number is a **multiple** of any number which is its factor. A number is the **common multiple** of two or more numbers, if each of these numbers is its factor. These factors need not belong to the same group. For instance, 9 and 6 are factors of 18, but they belong to different groups of numbers whose product is 18.

The number 360 is produced by the prime numbers $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$. But there are twenty-two numbers that, in different groups, will also produce 360; hence, 360 is a multiple of each of the twenty-two numbers.

*Example.*

$$\begin{aligned} 8a^2b^3c^2xy &= 2^3 \qquad a^2b^3c^2xy \\ 12a^4bcx &= 2^2 \cdot 3 \cdot a^4bcx \\ \underline{16a^3c^3x^2y^4} &= \underline{2^4 \qquad a^3c^3x^2y^4} \end{aligned}$$

Lowest common multiple $= 2^4 \cdot 3 \cdot a^4b^3c^3x^2y^4$

In the L. C. M. are found the different prime factors of the given numbers, each in its highest power. The lower powers are included in the highest power. If $a^4$ is a factor of a number, $a^3$, $a^2$, and $a$ are also factors of the number.

While $48\,a^4b^3c^3x^2y^4$ is the lowest multiple of the three numbers when associated, it is not the lowest multiple of two of them associated. The lowest multiple of each alone is itself.

### EXERCISE 24

Find the lowest common multiple:

1. $24\,x^3y^3z^4$, $30\,x^2y^4z^6$, $9\,x^2y^3z^4$, $21\,x^3y^6$
2. $5\,cd^2$, $7\,c^3dx$, $21\,cd^2x^5$
3. $a^2+7\,a+12$, $15(a^2+4\,a+4)$, $a^2+6\,a+8$
4. $a^2+4\,a+4$, $a^2+5\,a+6$, $(a^2-4)^2$
5. $a^2-4\,a+3$, $a^2+a-2$, $a^2-a-6$
6. $a^2+a-20$, $a^2-a-30$, $a^2-10\,a+24$
7. $1-3\,x+3\,x^2-x^3$, $(1-x)^4$, $1-2\,x+x^2$
8. $17(x^3+1)$, $34(x^3-1)$, $51(x^6+1)$, $68(x^{12}-1)$
9. $15(a-b)$, $21\,b^2(a^2-b^2)$, $35\,b^3(a+b)$
10. $10(x^2-x-12)$, $15(x-4)$, $9(x+3)$, $6(x^2-9)$
11. $27(x^2-ax+a^2)$, $3(x^2+ax+a^2)$, $81(x^4+a^2x^2+a^4)$

---

### INTERNS

**26.** An enclosed number, or a group of terms separated from other terms by marks of grouping, has the character and the relations of a single term or a single factor. It has a sign, + or −, a coefficient, — the numeral 1 if no other is expressed, — and when no exponent is given it has the exponent 1. If the intern is of the first power, the enclosing marks may be removed by multiplying the number by the coefficient with its sign.

Thus: $3(a-b)=3a-3b$; $-(a-b)=-a+b$.

If the intern has a numerical exponent, the number must be raised to the indicated power before it is multiplied by the coefficient. Thus:

$$-3(a-b)^2=-3(a^2-2ab+b^2)=-3a^2+6ab-3b^2.$$

### EXERCISE 25

Remove the enclosing marks:

1. $-5(m+n)^2$
2. $-3a(a^2-a-1)$
3. $+3ab(a^3-a^2-a)$
4. $+7(a-b)^3$
5. $+6(a+b)(a-b)$
6. $-1(a^3-4+m)$
7. $-11(1-3x-y+z)$
8. $-1(a^2-c-2)^2$
9. $-[-2a+3b-4c]$
10. $-[-2(a-b-c)]$
11. $-[a^2-b^2-3(x+y)]$
12. $+(-5[a+b]^2+1)$

### EXERCISE 26

Multiply, retaining the enclosing marks:

(a) $\overline{a+b}+c$ by $\overline{a+b}+c=\overline{a+b}^2+2\,\overline{a+b}\cdot c+c^2$.

(b) $(x+y)+c$ by $(x+y)+c=(x+y)^2+2(x+y)c+c^2$.

(c) $(a+x)+y$ by $(a+x)-y=(a+x)^2-y^2$.

1. $(a+b)+2$ by $(a+b)-2$
2. $\overline{a-b}-3$ by $\overline{a-b}+3$
3. $(a+b)+(c+d)$ by $(a+b)-(c+d)$
4. $x+(y+z)$ by $(m+n)$
5. $a+\overline{b+c}$ by $a-\overline{b+c}$
6. $(a+b)^3$ by $(a+b)$

7. $x^2 - x - y$ by $(x+y)$

8. $4+(2x-3)$ by $4-(2x-3)$

9. $(p+q)+r$ by $(p+q)-r$

10. $6ac(m^2+n^2)$ by $3a^2c^2(m^2-n^2)$

---

### EXERCISES IN EXPRESSION

**27.** 1. What is the area of a garden, if its length in yards is $a^2+4a+4$, and its breadth is $a+2$ yards?

2. How many cubic inches in a solid, if each of its dimensions is $2-b$ inches?

3. How many cubic inches in a solid, if its upper surface contains $a^2-ab+b^2$ square inches, and its height is $a+b$ inches?

4. A boy has $x+b$ marbles, another boy has $x-b$ marbles, and together they have 30 marbles. What number does $x$ represent?

5. A boy walks from his house 50 rods towards the east, then turns and walks 60 rods in the opposite direction. How far east of the house is he?

6. A girl is 14 years old; her sister is $-3$ years older. How old is the sister?

7. If steam will raise the temperature of a room $(5ab)°$, and incoming air will reduce the temperature $(9ab)°$ in the same time, what will be the result?

8. A given number is $100a+10b+c$. Represent another number of the same digits in reverse order.

9. If $x$ represents the number of minute spaces the hour hand of a watch passes over in one hour, what will represent the number of minute spaces the minute hand will pass over in the same time?

**10.** The minute hand of a watch passes over $ab$ spaces in a certain time; how many spaces will the hour hand pass over in the same time?

**11.** The length of a room is $c$ feet; the breadth is 3 feet less. What is the area of the ceiling in square feet?

**12.** Another room is 2 feet less in length than the one spoken of in Ex. 11, and it is 7 feet broader. What is the area of the floor?

**13.** The first of three horses cost $4b$ dollars; the second, three times as many dollars; the third, three times as many dollars as the second cost. What was the cost of the three horses?

**14.** A merchant bought three pieces of cloth at the rates of 7 yards for \$5, 6 yards for \$3, and 11 yards for \$9. Represent the cost of $x$ yards of the first piece, $y$ yards of the second piece, and $z$ yards of the third piece.

**15.** A freight train and an express train start at the same time from the same station. The rate of the freight train is 20 miles an hour; the rate of the express train is 50 miles an hour. How far apart will they be in $x + y$ hours, if they move in the same direction?

**16.** What is the length of a post, if $x$ feet in the mud, and $y$ feet in the water are two-thirds of its length?

**17.** A man can do a piece of work in $x$ days. How much of it can he do in one day? If the piece of work is making six pairs of shoes, how many pairs can he make in one day?

**18.** A and B start at the same time from two towns, and travel at the rates of $a$ miles and $b$ miles an hour. They meet in $x - 5$ hours. What is the distance from one town to the other?

19. A regiment is drawn up in solid square with $x$ men on a side. How many men in the regiment? Later, in the same regiment, there are but $x-5$ men on a side of the solid square. How many men remain?

20. If three numbers are consecutive, what is the difference between the first and the second? If the first number is $x$, what will represent each of the others? If the third number is $x$, what is the first?

21. A cistern holds $a$ gallons. In one hour one pipe pours in one-third of $a$ gallons; a second pipe pours in one-fourth of $a$ gallons; a third, one-sixth of $a$ gallons. What number of gallons will all the pipes together pour in, in one hour? In $5x$ hours? If there was a fourth pipe, what part should it fill in order that the cistern might be filled in one hour?

22. A fox is pursued by a dog. Two leaps of the dog equal 3 leaps of the fox. There are $a$ feet in one leap of the fox. How many feet are there in 3 leaps of the fox? In 2 leaps of the dog? In one leap of the dog?

23. A man went around the banks of a rice field $n$ times. He shot $br$ rice birds. What was the average number of birds for each time he went around the field?

24. A boy spent $b$ years in the grammar school, and $c$ years in the high school. In the same time his father paid $tx$ dollars for the support of public schools. How many dollars did the father pay annually?

25. There were $a$ companies, and $b$ men in each company. Each man used $c$ cartridges every week-day for $d$ weeks. How many cartridges were used?

26. There were $x+3$ girls, each of whom had $x$ books. A distribution of 2 more to each girl made how many in all?

# IV. Subtraction

**28.** The **difference** between two numbers is the number which added to the subtrahend will give the minuend.

(*a*)

From $5a$ take $12a$.

$12a + (-7a) = 5a$.

Remainder, $-7a$.

(*b*)

From $-b$ take $+b$.

$b + (-2b) = -b$.

Remainder, $-2b$.

(*c*)

From $3c$ take $-4c$.

$-4c + 7c = 3c$.

Remainder, $7c$.

(*d*)

From $-6d$ take $-2d$.

$-2d + (-4d) = -6d$.

Remainder, $-4d$.

### EXERCISE 27

What number must be added to

1. $-5c$ to make $-5c$?
2. $-dx$ to make $-3dx$?
3. $+6x$ to make $-x$?
4. $-8c$ to make $+11c$?
5. $+8c^2x$ to make $+17c^2x$?
6. $-9x^3$ to make $+21x^3$?
7. $-2ax^3$ to make $-ax^3$?
8. $+3x^3$ to make $-8x^3$?
9. $+7xy^3$ to make $-xy^3$?
10. $+17ab^2c$ to make $-ab^2c$?
11. $-15x^3z$ to make $+5x^3z$?
12. $+8a^2b$ to make $0$?
13. $-m^3x$ to make $-m^3x$?
14. $+33x^2y^3z$ to make $-x^2y^3z$?
15. $2x^2 + 7x + 3$ to make $7x^2 + 0 - 1$?
16. $a - b$ to make $3a - 5b + c$?

17. $16a^2 + 2ab$ to make $11a^2 - 5ab + 7bc$?

18. $2x^3 - 7x^2 - x + 6a^2$ to make $9x^3 - 7x^2 - 6x$?

19. $3n^3 + 2a^2 + 2$ to make $2a^2 - 5a - 3$?

20. $6a^2b$ to make $(a + b)^3$?

21. $2x^2 - 2x$ to make $(x - 1)^3$?

22. $-cd - ac$ to make 1?

23. $3a^2b - d$ to make 2?

24. $-3x^2 + 7x - 5$ to make 0?

### EXERCISE 28

1. Take $x^2 - ax + a^2$ from $2x^2 - 2ax + 3a^2$
2. Take $-4am + a + 11$ from $6am - 3bc + 11a$
3. Take $4x^3 - 2x^2 - 2x - 14$ from $7x^3 - 2x^2 + 2x + 2$
4. Take $x^2 + 2xy + 5xz - 3y^2$ from $x^2 - 3xy - y^2 + yz$
5. Take $2b^2 + 4ab - 3a^2$ from $4a^2 - 2ab + 3b^2$
6. Take $x^2 + 6x - 7$ from $3x^2 - 4x + 2$
7. Take $3x^3 + 4x^2y + 6xy^2 + 7y^3$ from $5x^2y + 4xy^2 + 7y^3$
8. Take $11a^2 - 11a + 11$ from $5a^2 - 7a + 2$
9. Take $2mn + 3xy - 13bc$ from 0
10. Take $-3x^2 + 11y - 3z$ from 0

### GROUPING TERMS

**29.** 1. In the number series, § 12, what is the opposite of $+5$? What is the opposite *of the opposite* of $+5$? The opposite *of the opposite* of $-5$?

2. Does $+5$ differ in unit value or in quality from $-(-5)$?

3. Is taking the opposite of a number the second time, or changing its quality the second time, restoring the number to its original state?

4. What is the effect of changing the quality of a number once, and indicating the second change?

### EXERCISE 29

Form a negative group from the last three terms:

Example (1) $a - b + c - d$

$a - (+b - c + d)$

Example (2) $a^3 + b^3 + a^2 + b^2 - a^4$

$a^3 + b^3 - (-a^2 - b^2 + a^4)$

1. $1 - 8xy + 15x^3y + 13x$
2. $y^2 - 10xy + 15x^2y^2 - 5$
3. $a + 2b - 3c + 4d + m$
4. $x - 3y + 2z - r + 8$
5. $14a + 15b + 8c - d + m$
6. $-x^2 - 2y^2 + 3z^2 + t - 11$
7. $3ax - ay + 5bx + by^2$
8. $4a^2c^2 + bc + bx + b^2y$
9. $9xz - yz^2 - xr^3 + yr$
10. $7a^3 - a^2b + ab^2 - b^3$

### EXERCISE 30

(a) $a^2 + ab + 5a + 5b$

$a(a + b) + 5(a + b)$

$(a + b)(a + 5)$

(b) $x^2y - xy^2 - xz^2 + yz^2$

$xy(x - y) - z^2(x - y)$

$(x - y)(xy - z^2)$

Group the following terms. Find a binomial factor in each group of two terms. Unite the monomial factors to form the second binomial factor.

1. $ax - ay + bx - by$
2. $ac + bc + ad + bd$
3. $xz - yz - xr + yr$
4. $a^3 - a^2b + ab^2 - b^3$
5. $x^3 + x^2y + xy^2 + y^3$
6. $a^3 - a^2y + ay^2 - y^3$
7. $bc - b^2 - ac + ab$
8. $2x^2 + 2xy - xz - yz$

9. $5a + 15b - 4ax - 12bx$

10. $7ay - 77y - 11ax + 121x$

11. $10ac - 35ad + 6bc - 21bd$

12. $12a^2 - 18ab + 8ac - 12bc$

## 30. MONOMIAL AND POLYNOMIAL FACTOR

Ex. (*a*) $8c^2x^2 - 12c^2d^2 + 4c^3$

$4c^2 \cdot 2x^2 + 4c^2 \times -3d^2 + 4c^2 \cdot c$

$4c^2(2x^2 - 3d^2 + c)$

Ex. (*b*) $51a^5 + 21a^4b^3 - 75a^3b^2$

$3a^3 \cdot 17a^2 + 3a^3 \cdot 7ab^3 + 3a^3 \times -25b^2$

$3a^3(17a^2 + 7ab^3 - 25b^2)$

### EXERCISE 31

Find the highest monomial factor, and prefix it to the polynomial factor enclosed:

1. $11m^8n^3c^3 - m^5n^6c^2$
2. $5ab^2 - 10a^2b - 5a^2b^2$
3. $57c^4d^2 - 38c^3d^3$
4. $9x^2y^3 - 6x^2y^2 + 3x^3y$
5. $m^3n - 21a^2bn + 30abn^3$
6. $a^3x^4y^2 - a^3x^3y^3 + a^3x^2y^4$
7. $7c^4d^2 + 14c^3d^3 + 7c^2d^2$
8. $3m^5n - 6m^4n - 45m^3n$
9. $5x^4y + 10x^3y^2 + 50x^2y^3 + 5xy^4$
10. $5n^3p - 10n^3q + 15n^3r$
11. $3a^2b + 3ab^2 + b^3$
12. $4a^3b + 6a^2b + 4a^2b^3$
13. $3a^3b - 21a^2b + 30ab$
14. $10a^4x^2 - 15a^2x^4 - 20a^3x^5$
15. $a^5b^2c + a^4b^3c + a^3b^4c$
16. $x^7y^2z + x^5y^4z + x^3y^6z$
17. $75x^4y^5 - 36x^2y^7z$

# V. Division

**31.** (*a*) $+90 \div (+30) = +3$ (*c*) $+84 \div (-21) = -4$

(*b*) $-76 \div (-19) = +4$ (*d*) $-26 \div (+13) = -2$

To determine the sign of the quotient consider the quality of the number by which the divisor was multiplied to make the dividend. If the dividend (or product) is negative, and the divisor (or given factor) is negative, what must be the sign of the quotient (or factor sought)?

Does the law of signs in division differ from the law of signs in multiplication?

(*e*)

$a^5 \div a^3 = a^{5-3}$, or $a^2$

$a^4 \div a^4 = a^{4-4}$, or $a^0$

$a^3 \div a^5 = a^{3-5}$, or $a^{-2}$

$a^0 \div a^6 = a^{0-6}$, or $a^{-6}$

$a^3 \div a^0 = a^{3-0}$, or $a^3$

(*f*)

$25\,a^3c^2n^4 \div 5\,a^5c^2d$, or

$25\,a^3c^2d^0n^4 \div 5\,a^5c^2d^1n^0$

Quotient: $5\,a^{-2}c^0d^{-1}n^4$, or

$5a^{-2}d^{-1}n^4$

(*g*)

$$a^4 \div a^4 = a^0;\quad a^4 \div a^4 = 1 \quad \therefore\ a^0 = 1$$

### EXERCISE 32

1. $-21\,a^4b^5 = -3\,a^2b^2 \times ?$
2. $+16\,a^3b^5c^7 = -2\,abc \times ?$
3. $-2\,abc^5d^4 = -ac^3d \times ?$
4. $+20\,m^2p^3q^4 = -4\,p^3 \times ?$
5. $-72\,a^7c^5n^2 = -8\,a^2n \times ?$
6. $+5\,d^9c^5n^4 = -5\,d^7c^2 \times ?$

7. $-36\,x^3y^2z^3 = -9\,xy^2z \times ?$

8. $(42\,a^3b^3c^2 - 7\,a^4bc^3) \div (-7\,a^3bc^2) = ?$

9. $(51\,x^3y^4z^3 - 34\,x^5y^2z^3) \div (-17\,x^2y^2z^3) = ?$

10. $(121\,m^3c + 22\,m^2c^5) \div 11\,m^2c = ?$

11. $(-24\,a^5x^2 + 36\,a^4x^3) \div (-6\,a^2x^2) = ?$

12. $(35\,m^3n^6 - 42\,m^4n^5) \div (-7\,m^2n^4) = ?$

13. $(15\,a^9x^{14}y - 12\,a^8x^{12}z) \div (-3\,a^7x^{10}) = ?$

14. $(-38\,a^5b^2xy^5 - 19\,a^5b^4x^2) \div (-19\,a^3bx) = ?$

## DIVISION OF POLYNOMIALS BY POLYNOMIALS

**32.** (a)

$$
\begin{array}{l|l}
a^3 + 3\,a^2b + 3\,ab^2 + b^3 & a + b \\ \cline{2-2}
\underline{a^3 + \phantom{3\,}a^2b} & a^2 + 2\,ab + b^2 \\
\quad + 2\,a^2b + 3\,ab^2 & \\
\quad \underline{+ 2\,a^2b + 2\,ab^2} & \\
\qquad\qquad + ab^2 + b^3 & \\
\qquad\qquad \underline{+ ab^2 + b^3} &
\end{array}
$$

(b)

$$
\begin{array}{l|l}
x^4 & x^3 + x^2 + x + 1 \\ \cline{2-2}
\underline{x^4 + x^3 + x^2 + x} & x - 1 + \dfrac{1}{x^3 + x^2 + x + 1} \\
\quad - x^3 - x^2 - x & \\
\quad \underline{- x^3 - x^2 - x - 1} & \\
\qquad\qquad\qquad + 1 &
\end{array}
$$

(c)

$$
\begin{array}{l|l}
x^8 - x^4y^4 + y^8 & x^4 - x^2y^2 + y^4 \\ \cline{2-2}
\underline{x^8 - x^6y^2 + x^4y^4} & x^4 + x^2y^2 - y^4 - \dfrac{+ 2\,x^2y^6 - 2\,y^8}{x^4 - x^2y^2 + y^4} \\
\quad + x^6y^2 - 2\,x^4y^4 + y^8 & \\
\quad \underline{+ x^6y^2 - \phantom{2\,}x^4y^4 + x^2y^6} & \\
\qquad\quad - x^4y^4 - x^2y^6 + y^8 & \\
\qquad\quad \underline{- x^4y^4 + x^2y^6 - y^8} & \\
\qquad\qquad\quad - 2\,x^2y^6 + 2\,y^8 &
\end{array}
$$

### TEST J

1. Why are the terms of the dividends in the given examples arranged according to the powers of $a$ or of $x$?

2. Should the divisors have similar arrangement?

3. How are the terms of the partial dividends arranged?

4. Why is it sufficient to divide the first term of the dividend by the first term of the divisor?

5. When may one term be divided by four terms or by any number of terms?

6. Why are the signs of the terms of the remainder (Ex. $c$) changed in the numerator of the fractional term of the quotient?

7. How may the examples be verified?

8. Write a rule for dividing one polynomial by another.

### EXERCISE 33

(d)

$$
\begin{array}{l|l}
x^6 - y^6 & x^4 - x^3y + xy^3 - y^4 \\ \cline{2-2}
\underline{x^6 - x^5y + x^3y^3 - x^2y^4} & x^2 + xy + y^2 \\
\quad + x^5y - x^3y^3 + x^2y^4 - y^6 & \\
\quad \underline{+ x^5y - x^4y^2 + x^2y^4 - xy^5} & \\
\qquad\qquad + x^4y^2 - x^3y^3 + xy^5 - y^6 & \\
\qquad\qquad \underline{+ x^4y^2 - x^3y^3 + xy^5 - y^6} &
\end{array}
$$

1. Divide $x^3 - 3x^2y + 3xy^2 - y^3$ by $x - y$
2. Divide $y^3 - 3y^2 + 3y - 1$ by $y^2 - 2y + 1$
3. Divide $a^4 + a^2b^2 + b^4$ by $a^2 - ab + b^2$
4. Divide $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ by $x^2 + 2xy + y^2$
5. Divide $a^8 + a^4b^4 + b^8$ by $a^4 + a^2b^2 + b^4$
6. Divide $a^{12} + b^{12}$ by $a^8 - a^4b^4 + b^8$

7. Divide $m^4 + n^4$ by $m^2 - 2mn + n^2$
8. Divide $27x^6 - 1$ by $3x^2 + 1$
9. Divide $16c^3 - 1$ by $8c^3 + 4c^2 + 2c + 1$
10. Divide $x^8 + 64$ by $x^4 - 4x^2 + 8$
11. Divide $x^7 - y^7$ by $x - y$

**EXERCISE 34**

Divide

1. $a^6 + 64$ by $a^4 - 4a^2 + 16$
2. $8a^3 - 27b^3$ by $4a^2 + 6ab + 9b^2$
3. $x^4 + 8x^3 + 25x^2 + 38x + 24$ by $x^2 + 3x + 4$
4. $x^4 + x^3 - x^2 + 9x + 30$ by $x^2 - 3x + 6$
5. $a^4 + 2a^3b - 13a^2b^2 + 34ab^3 - 15b^4$ by $a^2 + 5ab - 3b^2$
6. $4a^4 - 3a^2 + 7a - 12$ by $2a^2 - a + 3$
7. $5a^4 - 18a^3b + 5a^2b^2 - b^4$ by $b^2 + 3ab - a^2$
8. $1 - 8x + 17x^2 + 2x^3 - 24x^4$ by $1 - 5x + 6x^2$
9. $1 - 6x - 8x^2 + 6x^3 + 7x^4$ by $2x + x^2 + 1$
10. $120 - 26x + 19x^2 - 2x^3 + x^4$ by $x^2 + 12 + x$
11. $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$
12. $a^4 - 6a^2 + 25$ by $a^2 - 4a + 5$
13. $9x^4 + 20x^2 + 16$ by $3x^2 - 2x + 4$
14. $x^4 + 5x^3y + 2x^2y^2 - 10xy^3 - 8y^4$ by $x^2 - 2y^2$
15. $x^4 - 1$ by $x + 1$
16. $81x^4 - 16y^4$ by $3x^2 - 2y^2$
17. $a^4 - b^4$ by $a^3 + a^2b + ab^2 + b^3$
18. $32a^5 + b^5$ by $2a^2 + b^2$
19. $a^5 + a - 1$ by $a^3 + a^2 - 1$
20. $a^2 + 2ab + b^2 - c^2$ by $a + b + c$

### EXERCISES IN FORMING EQUALITIES

**33.** 1. There were 90 animals. There were $x$ camels, seven times as many monkeys as camels, and ten times as many horses as camels. How many times $x$ in the sum of all the animals? What is the numerical number for all the animals? Express the equality of these two numbers by using the sign of equality. Let the number containing $x$ come before the sign.

2. Some children gave a man with a bagpipe $x$ cents for each tune that he played. He had played five tunes when he was offered as many cents as the children had paid him if he would go away. He received 40 cents in all. How many times $x$ cents did he receive for playing? For ceasing to play? How many times $x$ cents did he receive in all? Express the equality of the algebraic number for cents paid and the numerical number for cents received.

3. A cowboy rode $x$ miles toward the mountains, and went back $y$ miles; then he was 30 miles from the ranch. What is the algebraic number for the miles from the ranch? Express the equality of the two numbers that stand for *miles distant from the ranch.* If the man had ridden $3x$ miles, and had gone back $5y$ miles, he would have been 50 miles from the ranch. Express a second equality of numbers standing for *miles distant from the ranch.*

4. The first of two lace-workers made $x$ yards of lace; the second made $y$ yards of lace. If each had made twice as many yards, the first would have made 6 yards more than the second made. What is the algebraic number for the difference? Express the equality of the two numbers that represent the difference.

5. A is twice as old as B. If B's age is $x$ years, represent the age of each man seven years ago. At that time, twice A's years and five times B's years were the same. Express an equality, and simplify the members of the equality.

6. Twenty-three times the number $x$ is just as much greater than 14 as 16 is greater than seven times $x$. Represent the two differences, and express their equality.

7. A room is $3x$ feet long. The width is two-thirds the length. If the length was 3 feet less, and the width was 3 feet more, the room would be square. Express an equality. (What is true of the sides of a square?)

8. The difference between two numbers is 25. If $x$ is the less number, what is the greater number? Represent the equality of three times the less, and the greater plus 5.

9. There were $x$ gallons in each of two casks. From the first, 80 gallons were drawn; from the second, 34 gallons. Three times the number of gallons left in the first cask equalled the number left in the second. Express the equality of the remainders.

10. A, B, and C together gave $76. A gave $x$ dollars. B gave as many as A and $10 more. C gave as many as A and B together. Represent their contributions. Add the three. What besides this number represents all the money contributed? Express an equality.

11. A father is four times as old as his son. If the son is $x$ years old, how old will he be in 24 years? How old will the father be in 24 years? The father will then be twice as old as his son. Express an equality.

12. Two numbers are to each other as 3 to 7, and $x$ is their highest common factor. Represent the numbers and their sum. The sum is also 140. Express an equality.

13. There are six persons, each of whom is four years older than the next younger. Let $x$ represent the age of the youngest. If the eldest is three times as old as the youngest, express an equality.

14. Divide the number 20 into two parts, such that the sum of three times one part and five times the other part shall be 84. Suppose $x$ to be one of the parts of 20; what is the other part? Find the sum of three times one part and five times the other part. What is the numerical expression for the sum? Express an equality.

15. Represent two numbers whose difference is 36. Represent their sum. Their sum is also twice their difference. Express an equality.

16. The difference between the squares of two consecutive numbers is 121. Let $x$ represent the less number. How much greater is the next consecutive number? Represent the difference of the squares. What given number also represents the difference of the squares? Express an equality.

17. A line is 2 feet and 4 inches in length. Divide the line into two parts, such that the length of one part shall be three-fourths of the length of the other. Let $x$ be the highest common factor of the length of the parts expressed in inches. Then $3x$ and $4x$ will represent the parts. What is the length of the line in terms of $x$? The length in inches? Express an equality.

18. There are of sheep and goats together 144. There are 7 sheep to every 5 goats. Represent the highest common factor of the number of sheep and the number of

goats. How many sheep? How many goats? How many in all? Express the equality of the algebraic expression and the given numerical expression for all in the flock.

**19.** One number exceeds another by 3. The square of the greater exceeds the square of the less by 99. If $x$ is the less number, what is the greater? Express the squares of the numbers, the difference of the squares, and an equality.

**20.** The sum of five numbers is 50. The numbers increase by 4. Express an equality.

**21.** In a family each son has as many sisters as he has brothers. If $x$ represents the number of sons, how many brothers has each son? How many sisters has each son? How many daughters are there? How many sisters has each daughter? If each daughter has twice as many brothers as she has sisters, express an equality.

**22.** Silk costs five times as much a yard as linen. Twenty-two yards of silk and fifty yards of linen cost $ 80. If the linen costs $x$ cents a yard, how many cents does a yard of silk cost? What will 50 yards of linen and 22 yards of silk cost? By the conditions of the problem, the whole cost is $80. In order to make an equality is it better to change $80 to cents or to divide the algebraic sum by 100?

**23.** A boat will float down a stream at the rate of $x$ miles an hour. A crew can row the boat without the help of the current at the rate of $y$ miles an hour. How many miles an hour can the crew and the current together take the boat down? How many miles does the current take from the rate of the crew when they row up the stream? It takes the crew 3 hours to row down $a$ miles, and 9 hours to row back. Express an equality.

**24.** If a wheel makes one revolution, its track is equal in length to the circumference of the wheel. Suppose the circumference of the wheel to be $x$ feet, how many revolutions will the wheel have made, if its track is 100 yards in length? Suppose another wheel, three feet greater in circumference, has made an equally long track, how many revolutions must it have made? Suppose the difference in the number of revolutions to be 5, and express an equality. Which wheel will make the greater number of revolutions in a given time? Should the circumferences of the wheels and the length of the tracks be given in the same denomination?

**25.** What is the circumference of a wheel which makes $y$ revolutions in going 400 feet? Of another wheel which makes 4 more revolutions in the same distance? If the sum of the circumferences of the two wheels is 45 feet, express an equality.

**26.** If 4 pounds of coffee at $y$ cents a pound, 3 pounds at $z$ cents, and 2 pounds at $y - z$ cents are mixed, what is the worth of one pound of the mixture? If 6 pounds of the mixture cost $m + n$ dimes, make an equality.

**27.** A can walk $m$ miles in $h$ hours. How many miles can he walk in one hour? B can walk $s$ miles in $r$ hours. How many miles can B walk in one hour? In 48 hours A can walk ten miles farther than B can walk. Express an equality.

# VI. The Equation

**34. Ratio.** — Two numbers considered with reference to the same unit of measure have a relation to each other called their ratio. The terms of the ratio are abstract numbers. The numbers may be the result of a measurement in which the unit is a unit of length, of surface, of weight, or of capacity; or the terms of the ratio may be the result of the measurements reduced by removing their highest common factor. Thus the length of a room may be 18 feet, the width 12 feet. The ratio of the length to the breadth is 18 : 12. But it may be given in lower terms, 9 : 6; or, in lowest terms, 3 : 2. In the last ratio, if 3 and 2 are considered direct results of measurement, the unit is 6 feet or 2 yards. A ratio is also expressed in the form of a fraction.

**35. The ratio of two unequal numbers** is changed when the numbers are subjected to the same operation, except when they are multiplied or divided by the same number.

| | | | |
|---|---|---|---|
| Given: $36 > 16$. | Ratio of the numbers, | | 9 : 4 |
| $36 + 2 > 16 + 2$ | " | " | 19 : 9 |
| $36 - 2 > 16 - 2$ | " | " | 17 : 7 |
| $36 \times 2 > 16 \times 2$ | " | " | 9 : 4 |
| $36 \div 2 > 16 \div 2$ | " | " | 9 : 4 |
| $(36)^2 > (16)^2$ | " | " | 81 : 16 |
| $\sqrt{36} > \sqrt{16}$ | " | " | 3 : 2 |

By means of inequalities, which are subject to changes not given above, results within limits are obtained. These results sometimes amount to a certainty. When, by the solution of a problem, it is found $x < 17$ and $x > 15$, there is no doubt of the numerical value of $x$ if it represents a whole number.

**36. The ratio of two equal numbers** is not changed when the members of the equality are subjected to the same operation.

| | |
|---|---|
| Given: $16 = 16.$ | Given: $a = a.$ |
| $16 + 2 = 16 + 2$ | $a + b = a + b$ |
| $16 - 2 = 16 - 2$ | $a - b = a - b$ |
| $16 \times 2 = 16 \times 2$ | $a \cdot b = a \cdot b$ |
| $16 \div 2 = 16 \div 2$ | $a \div b = a \div b$ |
| $(16)^2 = (16)^2$ | $a^3 = a^3$ |
| $\sqrt{16} = \sqrt{16}$ | $\sqrt[3]{a} = \sqrt[3]{a}$ |

In the six operations performed upon $16 = 16$ and $a = a$, the ratio of the numbers, $1:1$, is not changed; the equality is not destroyed. Upon this constancy of ratio the solution of a vast number of problems is based. The algebraic expression of a statement in the problem is equal to the algebraic expression of another statement or to a given numerical number. By performing upon each member of the expression of this equality, one or more of the operations performed upon $a = a$, a number required by the problem may be obtained.

The facts shown in the six transformations of the given equality, or identity, are not disputed; they are called self-evident truths, or *axioms*.

**Equations.** — An algebraic expression of the equality of two numbers is an *equation.*

(*a*) $2x + 2x = 4x$ (*b*) $5y = 15$

The first example is true whatever the value of $x$; the expression is simply an instance of "2 and 2 make 4." Such an expression of equality is an *identity.*

The second example is true only when $y = 3$; hence $5y = 15$ is an *equation of condition.* The word "equation" generally indicates equation of condition. If the value of the unknown number in an equation of condition be substituted for the unknown, as 3 for $y$ in $5y = 15$, the result will be an identity: $5 \times 3 = 15$, or $15 = 15$.

### EXERCISE 35

(*a*) $3 \times 2 = 6.$ $4 + 2 = 6.$

Compare $3 \times 2$ and $4 + 2$.

(*b*) $b = a.$ $c = a.$

Compare $b$ and $c$.

Is $b > c$?

$b < c$? $b = c$? Why?

Write a statement of the truth found by this comparison; also write a statement of each of the truths which appear in the six identities following $16 = 16$. Letter these statements, self-evident truths, or axioms, with print capitals. Letter the last, concerning numbers equal to the same number, E, the others in their order, A, S, M, D, P, R.

In writing solutions where space need not be given to the complete axiom, the appropriate letter will indicate the truth referred to.

## SOLUTION OF PROBLEMS

**37.** *Example A.* A man sold four times as many hats on Wednesday as he sold on Tuesday. On Tuesday he sold 10 more than he sold on Monday. He sold 80 hats in all. How many did he sell each day?

Let $x$ = the number of hats sold on Monday.

$x + 10$ = the number of hats sold on Tuesday.

$4x + 40$ = the number of hats sold on Wednesday.

Then $6x + 50$ = the whole number of hats sold,
but $80$ = the whole number of hats sold.

Hence, $6x + 50 = 80$ (1) Ax. ?

$6x = 30$ (2) Ax. ?

$x = 5$ (3) Ax. ?

$x + 10 = 15$, the number of hats sold on Tuesday.

$4x + 40 = 60$, the number of hats sold on Wednesday.

The facts of the problem are noted by using $x$ to represent one of the numbers sought, and by expressing the relations of $x$ in the other numbers.

The solution requires an equation. How may two expressions for the same quantity be found? It is stated that 80 hats were sold. The sum of the algebraic numbers for Monday, Tuesday, and Wednesday gives another number for all that were sold. Here then are materials for an equation (Ax. E).

The equation formed will be an equation of condition. The solution of the equation will show for what $x$ stands, or will show under what condition it serves in this problem.

The equation is to be solved for the value of $x$. Then $x$ must be made to stand alone, without coefficient, and without another number added to it or subtracted from it.

1. What may be done that $6x$ may stand alone? Erasing 50 will make the first number less than the second. What may be done to remove 50 and preserve the equality also?

2. State in a complete sentence the truth that warrants the removing of a term from a member of the equation.

3. When $6x$ stands alone, what may be done to allow $x$ to stand alone? Give the axiom that warrants the removal of the coefficient of $x$.

4. Why write, "Let $x =$ the number of hats sold"? Why not say, "Let $x =$ hats sold"? When a person solves the problem, does he expect to find the value of $x$ to be a number or some hats?

5. What advantage is there in careful arrangement, such as bringing signs of equality under one another? What advantage is there in numbering the equations?

*Example B.* A post is one-fourth of its length in mud, one-third of its length in water, and ten feet of its length are out of water. What is its length?

Let $x =$ the number of feet in the length of the post.

$\frac{x}{4} =$ the number of feet in the mud.

$\frac{x}{3} =$ the number of feet in the water.

$\frac{x}{4} + \frac{x}{3} + 10 =$ the number of feet in the length of the post.
$x =$ the number of feet in the length of the post.

$$x = \frac{x}{4} + \frac{x}{3} + 10 \qquad (1) \qquad \text{Ax. ?}$$

$$12x = \frac{12x}{4} + \frac{12x}{3} + 120 \qquad (2) \qquad \text{Ax. ?}$$

$$12x = 3x + 4x + 120 \qquad (3)$$

$$5x = 120 \qquad (4) \qquad \text{Ax. ?}$$

$$x = 24 \qquad (5) \qquad \text{Ax. ?}$$

$$\frac{x}{4} = 6. \qquad \frac{x}{3} = 8.$$

The length of the post is twenty-four feet.
Verification: Substitute 24 for $x$ in Equation (1).

$$24 = \frac{24}{4} + \frac{24}{3} + 10$$

$$24 = 24$$

An identity results; hence 24 satisfies the equation.

### TEST K

1. In Example $B$ what is the object in multiplying both members of equation (1)?

2. What is the authority for the multiplying?

3. Would 13 answer for a multiplier? Would 24? Why select 12?

4. What is there in the problem that suggests measuring the post in feet?

5. If the post were measured in inches, what number should be in the equation in place of 10? If the post were measured in yards, what number should be in place of 10?

6. If there were in the statement, "Let $x =$ the length," and $x$ was found to be 24, what would be the length of the post?

7. Does $x$ stand for things or for a number? If for a number, does it matter what the unit is?

8. Should $x$ represent in the statement what cannot be had in the result?

9. What axioms warrant equations (1), (2), (4), (5)?

10. In the change from equation (2) to equation (3) is there a change of form? Is there a change of value? Why is there no reference to an axiom?

*Example C.* A said to B, "If three-fourths, three-tenths, and four-fifths of my age were added to my age, the sum would be 3 less than three times my age." What was A's age?

Let $x =$ A's number of years.

$$x + \frac{3x}{4} + \frac{3x}{10} + \frac{4x}{5} = 3x - 3 \qquad (1) \quad \text{From the conditions.}$$

$$20x + \frac{20 \cdot 3x}{4} + \frac{20 \cdot 3x}{10} + \frac{20 \cdot 4x}{5} = 60x - 60 \qquad (2) \quad \text{Ax. M}$$

$$20x + 15x + 6x + 16x = 60x - 60 \qquad (3)$$

$$57x = 60x - 60 \qquad (4)$$

$$57x - 60x = -60 \qquad (5) \quad \text{Ax. S}$$

$$-3x = -60 \qquad (6)$$

$$x = 20 \qquad (7) \quad \text{Ax. D}$$

A's age was twenty years.

Verification: Substitute 20 for $x$ in equation (1).

$$20 + \frac{3 \cdot 20}{4} + \frac{3 \cdot 20}{10} + \frac{4 \cdot 20}{5} = 3 \cdot 20 - 3$$

$$20 + 15 + 6 + 16 = 60 - 3$$

$$57 = 57$$

An identity results; hence 20 satisfies the equation. The equation of condition, equation (1), is true if $x = 20$.

*Example D.* G is 25 years older than F; and G's age is as much greater than 20 years as F's age is less than 85. What is the age of each?

Let $x =$ the number of G's years,

$x - 25 =$ the number of F's years.

$x - 20 =$ G's years less 20 years.

$85 - (x - 25) =$ 85 years less F's years.

$$85 - (x - 25) = x - 20 \qquad (1) \quad \text{From the conditions.}$$

$$85 - x + 25 = x - 20 \qquad (2)$$

$$-x - x = -20 - 85 - 25 \qquad (3) \qquad \text{Ax. S}$$

$$-2x = -130 \qquad (4)$$

$$x = 65 \qquad (5) \qquad \text{Ax. D}$$

$$x - 25 = 40 \qquad (6)$$

By reversing the members of equation (1).

$$x - 20 = 85 - (x - 25) \qquad (1)$$

$$x - 20 = 85 - x + 25 \qquad (2)$$

$$x + x = 85 + 25 + 20 \qquad (3) \qquad \text{Ax. A}$$

$$2x = 130 \qquad (4)$$

$$x = 65 \qquad (5) \qquad \text{Ax. D}$$

$$x - 25 = 40 \qquad (6)$$

Verification: Substitute 65 for $x$ in equation (1).

$$65 - 20 = 85 - (65 - 25)$$

$$45 = 45$$

An identity is produced; hence 65 satisfies the equation. The equation of condition is true if $x = 65$.

### TEST L

1. Why is the unknown quantity represented by $x$? Would an interrogation point serve as well?

2. In example ($A$) is the number of hats sold on Tuesday any better known than the number sold on Monday? Why or why not?

3. If $x$ represented the number sold on Tuesday, how would the number sold on Monday be represented? The number sold on Wednesday?

4. Let $x$ represent the number sold on Wednesday. How could the numbers for the other two days be represented?

5. In the given example, $x = 5$; what should be the value of $x$ when $x$ stands for the number sold on Tuesday? For the number sold on Wednesday?

6. In example ($D$), the equation is solved twice? Is it lawful to change an equation, member for member? To reverse the members of an inequality?

7. In example ($D$), after the members of equation (1) are reversed, what is the change in the equations following?

8. If $-x$ is a term of the second member, what may be added to that member to produce 0 in place of $-x$? If nothing further were done, would the equation become an inequality? Which member of the inequality would be the greater?

9. How many operations may be performed upon both members of an equation, at the same time, without affecting the equality of the members?

10. In example ($A$), 50 is taken from both members of an equation; in example ($B$), both members are multiplied by 12; in example ($D$), 20 is added to both members. In each of these cases the members are made greater or less. Does the process of solving an equation depend upon the value of the members? Explain.

11. Does ratio depend upon the magnitude of the terms which express it? A table is measured in inches, and the ratio of the length to the breadth is 72 : 36. In lower terms the ratio is 6 : 3; in lowest terms it is 2 : 1. By what unit might the table be measured to give 6 : 3? To give 2 : 1? Do these transformations of the expression of the ratio alter the relation of the length to the breadth as at first expressed?

12. Does a ratio hold so long as each of its transformations, when reduced to the lowest terms, gives the same numbers that the original gives?

13. Does an equation hold so long as its members maintain the ratio $1:1$; that is, so long as substituting the same number for $x$ will change the equation to an identity, whatever may have been its lawful transformation?

14. What is the condition under which $2x = 130$? Could $5x = 130$ also be true in the same problem?

15. Could an identity become an equation of condition? Take an identity containing $x$, as

$$(x + 5)(x + 25) = (x + 15)^2 - 100,$$

make any of the transformations warranted by the axioms, and endeavor to obtain therefrom a value for $x$.

**EXERCISE IN THE CONSTRUCTION OF PROBLEMS. I.**

**38.** Convert numbers 5–23, § 33, Exercises in Forming Equalities, into complete problems.

Omit the use of "$x$" and the direction "represent the equality." The symbol $x$ and the equation are aids to the solution of a problem which is stated in words.

Each of the exercises 5–23, § 33, contains the three elements of a problem leading to a single equation of the first degree:

1. Reference to an unknown number;
2. Conditions from which an algebraic expression can be formed when the unknown number has been represented by $x$;
3. Further conditions from which can be formed a second algebraic expression equal to the first; or, a numerical number which is equal to the first algebraic expression.

# VII. Fractions

**39.** (*a*) Given, $420 \div 14 = 30$.

1. $(420 \times 2) \div 14 = ?$
2. $420 \div (14 \div 2) = ?$
3. $(420 \div 2) \div 14 = ?$
4. $420 \div (14 \times 2) = ?$
5. $(420 \times 2) \div (14 \times 2) = ?$
6. $(420 \div 2) \div (14 \div 2) = ?$

**TEST M**

1. What operations upon dividend or upon divisor will affect the quotient? How?

2. What operations will not affect the quotient?

Given, $$\frac{2 \cdot 2 \cdot 3 \cdot 5 \cdot 7}{2 \cdot \quad 7} = 2 \cdot 3 \cdot 5$$

The factors of the dividend not in the divisor also are the factors of the quotient.

3. Remove factor 2 from both dividend and divisor. What change in the quotient follows? Remove 7 also. What change follows?

4. Introduce $-5$ into each term. Is the quotient changed?

5. Introduce $4a^2bc$ into each term, at the same time remove 2 and 7 from each term. What is the effect?

6. What effect has multiplying both dividend and divisor by $+1$? By $-1$?

7. Will squaring both dividend and divisor affect the quotient? Will taking the square root?

8. Add 4 to 420 and to 14. Divide. Will the quotient be greater or less than the given quotient? Subtract 4 from each term. How is the quotient affected?

9. Is it safe to say without further trial that the operations in Example (*a*), 5 and 6, are the only operations that will not change the quotient?

10. By what may the numerator and by what may the denominator of $\frac{3}{4}$ be multiplied to invert the fraction? When 3, a dividend, is multiplied is the quotient changed? When 4, a divisor, is multiplied, how is the quotient changed?

11. Which is greater, $\frac{a}{b}\times 1$ or $\frac{a}{b}\times\frac{2}{2}$? $\frac{a}{b}\times 1$ or $\frac{a}{b}\times\frac{n^2x}{n^2x}$.

(*b*) Given $\frac{14}{420}=\frac{1}{30}$

1. $\frac{14\times 2}{420}=\frac{2}{30}$
2. $\frac{14}{420\div 2}=\frac{2}{30}$
3. $\frac{14}{420\times 2}=\frac{1}{60}$
4. $\frac{14\div 2}{420}=\frac{1}{60}$
5. $\frac{14}{420}\times\frac{2}{2}=\frac{1}{30}$
6. $\frac{14\div 2}{420\div 2}=\frac{1}{30}$

(*c*) Given 420 : 14, or 30 : 1

1. 840 : 14, or 60 : 1
2. 420 : 7, or 60 : 1
3. 840 : 28 or 30 : 1
4. 210 : 7, or 30 : 1

If a fraction is a representation of a division in which a quotient is sought, if it is the representation of a quotient, or if it is the ratio of two numbers, the quotient arising from the division, the value of the fraction, and the ratio of the numbers will not be altered by multiplying both the given terms or dividing both the given terms by the same number. That is, when the relation of the dividend and divisor, of the numerator and the denominator, or of the terms of the ratio, is not changed, the value is not affected.

**40.** A fraction may be transformed for the purpose of changing its denomination or to reduce to lower terms.

*Example* 1. Reduce $\frac{12\,a^2bc^3d}{18\,a^3b^2cd^2x}$ to its lowest terms.

$$\frac{12\,a^2bc^3d}{18\,a^3b^2cd^2x} \div \frac{6\,a^2bcd}{6\,a^2bcd} = \frac{2\,c^2}{3\,abdx};$$

or,
$$\frac{6\,a^2bcd \times 2\,c^2}{6\,a^2bcd \times 3\,abdx} = 1 \times \frac{2\,c^2}{3\,abdx} = \frac{2\,c^2}{3\,abdx}.$$

*Example* 2. Reduce $\frac{2\,a^2b^2 - 8\,b^2}{6\,b^3(a^2+3\,a-10)}$ to its lowest terms.

$$\frac{2\,a^2b^2 - 8\,b^2}{6\,b^3(a^2+3\,a-10)} = \frac{2\,b^2(a-2) \times (a+2)}{2\,b^2(a-2) \times 3\,b(a+5)}$$

$$= 1 \times \frac{a+2}{3\,b(a+5)} = \frac{a+2}{3\,b(a+5)}$$

A fraction is reduced to its lowest terms by dividing the numerator and the denominator by their highest common factor, or by removing the factors common to the numerator and denominator. The latter method is equivalent to separating the fraction into two fractions, one of which is equal to 1.

*Example* 3. Change $\frac{ab}{2xy}$, $\frac{3c}{5yz}$, $\frac{6m}{10x^2yz}$, to other equivalent fractions having for denominator $30ax^2yz$.

$$30ax^2yz = 2xy \times ? \qquad \frac{ab \quad \times 15axz}{2xy \quad \times 15axz} = \frac{15a^2bxz}{30ax^2yz}.$$

$$30ax^2yz = 5yz \times ? \qquad \frac{3c \quad \times 6ax^2}{5yz \quad \times 6ax^2} = \frac{18acx^2}{30ax^2yz}.$$

$$30ax^2yz = 10x^2yz \times ? \qquad \frac{6m \quad \times 3a}{10x^2yz \times 3a} = \frac{18am}{30ax^2yz}.$$

*Example* 4. Change $\frac{13ab}{21b^2x^2y^3}$, $\frac{11a}{7bxy}$, $\frac{5a^4}{6xy^2}$ to equivalent fractions having the lowest common denominator.

$$42b^2x^2y^3 \div 21b^2x^2y^3 = 2$$

$$42b^2x^2y^3 \div 7bxy = 6bxy^2$$

$$42b^2x^2y^3 \div 6xy^2 = 7b^2xy$$

$$\frac{13ab \quad \times 2}{21b^2x^2y^3 \times 2} = \frac{26ab}{42b^2x^2y^3}$$

$$\frac{11a \quad \times 6bxy^2}{7bxy \times 6bxy^2} = \frac{66abxy^2}{42b^2x^2y^3}$$

$$\frac{5a^4 \times 7b^2xy}{6xy^2 \times 7b^2xy} = \frac{35a^4b^2xy}{42b^2x^2y^3}$$

The lowest common denominator of two or more fractions is the lowest common multiple of their denominators. When the L. C. D. cannot be determined by inspection, it may be found by factoring as in example 5.

*Example* 5. $\dfrac{3a+7}{a^2-6a+9}, \quad \dfrac{25}{a^2-4a+4}, \quad \dfrac{3cd}{3a^2-9a}.$

$$a^2-6a+9 = (a-3)^2$$
$$a^2-4a+4 = (a-2)^2$$
$$3a^2-9a = 3a(a-3)$$
$$\text{L. C. M.} = 3a(a-2)^2(a-3)^2$$

(1) $$\frac{3a(a-2)^2(a-3)^2}{(a-3)^2} = 3a(a-2)^2$$

(2) $$\frac{3a(a-2)^2(a-3)^2}{(a-2)^2} = 3a(a-3)^2$$

(3) $$\frac{3a(a-2)^2(a-3)^2}{3a\ \ (a-3)} = (a-2)^2(a-3)$$

$$\frac{(3a+7)\quad \times 3a(a-2)^2}{(a^2-6a+9)\times 3a(a-2)^2} = \frac{(9a^2+21a)(a-2)^2}{3a(a-2)^2(a-3)^2}$$

$$\frac{25\quad \times 3a(a-3)^2}{(a^2-4a+4)\times 3a(a-3)^2} = \frac{75a(a-3)^2}{3a(a-2)^2(a-3)^2}$$

$$\frac{3cd\quad \times (a-2)^2(a-3)}{(3a^2-9a)\times (a-2)^2(a-3)} = \frac{3cd(a-2)^2(a-3)}{3a(a-2)^2(a-3)^2}$$

### EXERCISE 36

Reduce the following fractions to equivalent fractions having the lowest common denominator:

1. $\dfrac{3}{4a}, \quad \dfrac{5}{12a^3}, \quad \dfrac{4}{6a^3}$

2. $\dfrac{b}{x-b}, \quad \dfrac{bx}{b^2-x^2}, \quad \dfrac{b^2}{x^2-b^2}, \quad \dfrac{x}{b-x}$

3. $\frac{c}{c-b}$, $\frac{bc}{c^2-b^2}$, $\frac{b^2}{c^2+b^2}$, $\frac{b}{c+b}$

4. $\frac{5}{a^2-1}$, $\frac{1}{a-1}$, $\frac{3}{a+1}$, $\frac{4}{(a+1)^2}$, $\frac{x}{(a-1)^2}$

5. $\frac{bx}{x^3-b^3}$, $\frac{b+x}{x^2+bx+b^2}$, $\frac{b}{x-b}$

## EXERCISE 37

Write in lowest terms:

1. $\frac{x^2+xy}{2\,xy}$

2. $\frac{x^2+xy}{x^2-xy}$

3. $\frac{x^3+a^3}{x^2-a^2}$

4. $\frac{4\,(x+a)^2}{5\,(x^2-a^2)}$

5. $\frac{5\,a^2x-15\,ay^2}{10\,a^2x}$

6. $\frac{x^2+3\,x+2}{x^2+6\,x+5}$

7. $\frac{a^2-12\,a+35}{a^2-2\,a-15}$

8. $\frac{c^2-2\,c-15}{c^2+10\,c+21}$

9. $\frac{c^2+5\,c-24}{c^2-10\,c+21}$

10. $\frac{a^2+9\,a-36}{a^2+6\,a-27}$

11. $\frac{x^3+n^3}{x^4+n^2x^2+n^4}$

12. $\frac{x^4-1}{x^6-1}$

13. $\frac{x^6+b^2x^3y}{x^6-b^4y^2}$

14. $\frac{x^4+b^2x^2+b^4}{x^6-b^6}$

15. $\frac{x^2-2\,ax+a^2}{x^3-3\,x^2a+3\,xa^2-a^3}$

16. $\frac{3\,x^3y-9\,x^2y}{x^2-8\,x+15}$

17. $\frac{x^2+5\,x+6}{x^2-4\,x-12}$

18. $\frac{8\,x^3+27\,y^3}{9\,y^2+12\,xy+4\,x^2}$

19. $\frac{a^8+b^8}{a^{16}+b^{16}}$

20. $\frac{a^4-b^4}{a^4-2\,a^2b^2+b^4}$

**EXERCISE 38**

Write with lowest common denominator:

1. $\frac{a+b}{a^2-b^2}, \frac{15a^3c}{3a^2-3ab}, \frac{4a^2-4b^2}{a^2-2ab+b^2}, \frac{1}{3a^2m-3b^2m}$

2. $\frac{27x^3y}{3x^2+6xy+3y^2}, \frac{a+b}{(x+y)^3}, \frac{3b^2c}{b^3(x^2-y^2)}$

3. $\frac{a^4+a^2b^2+b^4}{a^3+b^3}, \frac{74(a-b)^2}{37(a^2-b^2)}, \frac{5a^3+5ab^2}{(a+b)^3}$

4. $\frac{m^3-n^3}{m^2+mn+n^2}, \frac{m^2-2mn+n^2}{(m-n)^3}, \frac{3(m+n)(a+b)}{(m+n)^2}$

5. $\frac{a^4+b^4}{a^8+2a^4b^4+b^8}, \frac{a^4-b^4}{5x(a^8-b^8)}, \frac{15a^2b^5c}{45ab^3x}$

6. $\frac{a+3}{a^2+5a+6}, \frac{a+1}{a^2+3a+2}, \frac{a-1}{a^2+2a-3}$

7. $\frac{a}{b}, \frac{a^3b}{ab-b^2}, \frac{ab^2-b^3}{a^3-b^3}$

8. $\frac{1}{(a-b)(b-c)}, \frac{1}{(c-d)(d-a)}, \frac{1}{15a}$

9. $\frac{9}{16-b^2}, \frac{4+b}{8+2b}, \frac{4-b}{16-8b+b^2}$

10. $\frac{1}{a-2x}, \frac{1}{a^2-5ax+6x^2}, \frac{1}{a-3x}$

**TEST N**

1. What is the effect of multiplying the first term of a ratio? Of dividing the second term? Of dividing both terms by the same number?

2. Is a fraction a representation of a product or of a quotient? Is a ratio a product or a quotient?

3. What two transformations of a fraction are possible without changing the value of the fraction?

4. What is the value of a fraction whose numerator and denominator are identical?

5. What is meant by lowest common denominator? By lowest terms?

6. Are there two numbers whose ratio would not be changed by the addition of the same number to each or by the subtraction of the same number from each?

7. Can fractions be changed to *equivalent* fractions having a common numerator?

8. In changing fractions to others having a common denominator, why choose the lowest?

9. If the denominators of three fractions were $a$, $b$, and $c$, what is the only limit or condition in the choice of a common denominator?

10. Given $3\,ad + \frac{ab}{b}$. The fraction has a sign. As the numerator and the denominator have no sign expressed, what sign has each?

11. Change two of the three signs as many times as possible, and after each change divide. Is the quotient the same as the first quotient, $+(+a)$? Does $+(+a)$ occur more than once? What changes give $-(-a)$? What is the difference between $+(+a)$ and $-(-a)$?

12. Write the law of signs observed in this division.

13. $\frac{12}{27} \times \frac{-1}{-1} = \frac{-12}{-27}$. By this multiplication are the numbers changed numerically? How? Is the value of the fraction changed?

14. $\frac{2 \times 6}{3 \times 5 \times 7}$. Is the value of the fraction changed by changing the sign of 7? Of 3 and of 7? Of 3 and of 7, and of 2 and of 6? Of 2 only? Of 3, of 5, and of 7? By changing the sign of one factor in the numerator and any one in the denominator, at the same time?

15. Given,

$$+\frac{-a \times -(a+b)}{(a-b) \times (a+5)} \text{ and } -\frac{a \times (a+b)}{-(a-b) \times (a+5)}.$$

What is the difference in the value of the two forms?

16. Changing the signs of how many factors will change the sign of the product?

17. If it is desirable in obtaining the least common denominator to change the sign of one or more factors in the denominator, so that the sign of the denominator as a whole is changed, what one of two changes must be made to preserve the value of the fraction?

18. Write a rule for examples such as example 4, page 56.

19. Given, $\frac{16\,a^4b^2c}{20\,a^3b^3d}$. Reduce the given fraction to its lowest terms. Multiply it by $5\,x$. Divide it by $5\,x$. Multiply both numerator and denominator by $5\,x$. If the given fraction represents division, what is the quotient? If it represents a ratio, write it in the form of a ratio in the lowest terms.

20. Is it well to reduce to lowest terms before reducing to L. C. D.?

**EXERCISE 39**

**41.** Combine the given fractions:

*Example* 1. $\frac{a}{3}+\frac{b}{3}=\frac{a+b}{3}$

*Example* 2. $\frac{3}{c}-\frac{3}{x}=\frac{3x}{cx}-\frac{3c}{cx}=\frac{3x-3c}{cx}$

*Example* 3. $\frac{a}{a-b}-\frac{b}{a+b}=\frac{a(a+b)}{(a-b)(a+b)}-\frac{b(a-b)}{(a-b)(a+b)}$

$$\frac{a^2+ab}{a^2-b^2}-\frac{ab-b^2}{a^2-b^2}=\frac{a^2+b^2}{a^2-b^2}$$

1. $\frac{x-5y}{5}+\frac{3x-y}{5}$
2. $\frac{c-3d}{4}-\frac{c-3d}{3}$
3. $\frac{4}{1-a}-\frac{5}{1-a^2}$
4. $\frac{n}{3+n}+\frac{n}{n^2-9}$
5. $\frac{6x-5a}{2}-\frac{4x-7a}{3}$
6. $\frac{2a-3}{12}-\frac{a+2}{9}+\frac{5a+8}{6}$
7. $\frac{5}{a+x}+\frac{8}{a-x}-\frac{3x}{a^2-x^2}$
8. $\frac{x+5}{x-5}-\frac{x-5}{x+5}-\frac{19x+5}{x^2-25}$
9. $\frac{a+1}{a-1}+\frac{a-1}{a+1}$
10. $\frac{2}{(a+b)^2}+\frac{1}{a+b}$
11. $\frac{n^2}{x-z}-\frac{mn}{x+z}$
12. $\frac{4}{c^2+x^2}-\frac{7}{c^2-x^2}$
13. $\frac{x-3}{5x}+\frac{x^2-9}{10x^2}-\frac{8-x^3}{15x^3}$
14. $\frac{a-x}{x}+\frac{a+x}{a}-\frac{a^2-x^2}{2ax}$
15. $\frac{a+2b}{a-2b}-\frac{a-2b}{a+2b}$
16. $\frac{2}{xy}-\frac{3y^2-x^2}{xy^3}$

*Example* (*A*). $\dfrac{x+4}{x^2+x+1}-\dfrac{1}{x-1}-\dfrac{x^2+4x-2}{1-x^3}$

$$\frac{(x+4)(x-1)}{x^3-1}-\frac{x^2+x+1}{x^3-1}+\frac{x^2+4x-2}{x^3-1}$$

$$\frac{x^2+3x-4-x^2-x-1+x^2+4x-2}{x^3-1}=\frac{x^2+6x-7}{x^3-1}$$

or, $\dfrac{4+x}{1+x+x^2}+\dfrac{1}{1-x}-\dfrac{x^2+4x-2}{1-x^3}$

$$\frac{(4+x)(1-x)}{1-x^3}+\frac{1+x+x^2}{1-x^3}-\frac{x^2+4x-2}{1-x^3}$$

$$\frac{4-3x-x^2+1+x+x^2-x^2-4x+2}{1-x^3}=\frac{7-6x-x^2}{1-x^3}$$

In example (*A*) the L. C. M. of the first two terms is $x^3-1$. The third denominator is given the same form by changing the sign of the fraction and the sign of the denominator as a whole: $-(1-x^3)=x^3-1$. In the second solution, similar changes are made in the second fraction, the order of the terms in the first denominator is reversed, and the fractions are reduced to others having $1-x^3$ for their L. C. D. If the terms of the second result are multiplied by $-1$, the two results will have the same form.

**17.** $\dfrac{1}{1+x}-\dfrac{1}{1-x^3}+\dfrac{1}{1-x+x^2}$

**18.** $\dfrac{2}{a^3-b^3}-\dfrac{1}{b-a}+\dfrac{3}{a^2+ab+b^2}$

**19.** $\dfrac{1}{(x+1)(x-1)}-\dfrac{2}{x^2+1}-\dfrac{1}{(1-x^2)(1+x^2)}$

**20.** $\dfrac{1}{x^2-ax+a^2}-\dfrac{1}{a^3+x^3}-\dfrac{3}{x+a}$

**21.** $\dfrac{2}{b^2-3b+2}+\dfrac{2}{b^2-b-2}-\dfrac{1}{b^2-1}$

*Ex.* (*B*). $\dfrac{1}{1+b^4}+\dfrac{1}{1+b^2}+\dfrac{1}{1+b}+\dfrac{1}{1-b}$

$$\frac{1}{1-b}+\frac{1}{1+b}=\frac{2}{1-b^2}$$

$$\frac{2}{1-b^2}+\frac{1}{1+b^2}=\frac{3+b^2}{1-b^4}$$

$$\frac{3+b^2}{1-b^4}+\frac{1}{1+b^4}=\frac{4+b^2+2b^4+b^6}{1-b^8}$$

*Ex.* (*C*). $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}$

$$\frac{1}{(a-b)(a-c)}-\frac{1}{(b-c)(a-b)}+\frac{1}{(a-c)(b-c)}$$

22. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$

23. $\dfrac{a^2}{(a-b)(a-c)}-\dfrac{b^2}{(b-a)(c-b)}-\dfrac{c^2}{(c-a)(b-c)}$

24. $\dfrac{a+c}{(a-b)(b-c)}-\dfrac{b+c}{(c-a)(b-a)}-\dfrac{a+b}{(a-c)(c-b)}$

25. $\dfrac{x^2-yz}{(x-y)(x-z)}+\dfrac{y^2+xz}{(y-x)(y+z)}+\dfrac{z^2+xy}{(z-x)(z+y)}$

26. $\dfrac{x-2}{(x-1)(x-3)}+\dfrac{x-1}{(x-2)(3-x)}+\dfrac{x-3}{(1-x)(2-x)}$

27. $\dfrac{1}{a(a-c)(a-x)}+\dfrac{1}{c(c-a)(c-x)}+\dfrac{1}{x(x-c)(x-a)}$

28. $\dfrac{1}{x^4+y^4}+\dfrac{1}{x^2+y^2}+\dfrac{1}{x+y}+\dfrac{1}{x-y}$

## EXERCISE 40

**42.** Reduce the mixed numbers to the form of a fraction:

*Example*

$$a^2-3a-\frac{3a(3-a)}{a-2}, \text{ or } (a^2-3a)\times\frac{a-2}{a-2}-\frac{3a(3-a)}{a-2}$$

$$\frac{a^3-5a^2+6a}{a-2}-\frac{9a-3a^2}{a-2}=\frac{a^3-2a^2-3a}{a-2}$$

1. $a^5+7a+2-\frac{3a-2}{5}$
2. $2y+x+1+\frac{5y^2}{x-3y}$
3. $\frac{a^2-ay+y^2}{x+a}-a+x+y$
4. $\frac{x^4}{1-x}+x^3+x^2+x+1$
5. $1+2a+4a^2-\frac{a^3+1}{2a-1}$
6. $\frac{x^4+2x^2+1}{x^2+x+1}-(x-x^2-1)$
7. $\frac{8x^2+7y^2}{3x-2y}-2x-3y$
8. $2a-4x-\frac{6a^2-9x^2}{4a+3x}$
9. $\frac{15x^2-8y^2}{5x-3y}-3x-4y+5$
10. $\frac{a^3-1+3a}{a^2-a+1}+a+1$
11. $a^4-4-\frac{3a^2}{a^4+1}$
12. $2x-a-\frac{a-3x}{4}$
13. $a^2-ax+x^2-\frac{ax^3+x^4}{(a+x)^2}$
14. $a^3-a^2+1-\frac{-a^3-a^2+1}{3}$
15. $5-\frac{n^3-n^2-n-1}{3xy}$
16. $8m^2-\frac{a^2-2ab+b^2}{1-m}$
17. $3a+1-\frac{(-m^2-a)^2}{3a}$
18. $(a+b)^2-\frac{(a+b)^2}{(a+b)^3}$

**EXERCISE 41**

**43.** Perform the multiplication indicated, and reduce the product to the simplest form:

*Ex.* (a). $\dfrac{a+b}{a^2-b^2}\times\dfrac{(a-b)^2}{(a+b)^2}$

$$\frac{(a+b)\times(a-b)(a-b)}{(a+b)(a-b)\times(a+b)(a+b)}=\frac{(a+b)(a-b)}{(a+b)(a-b)}\times\frac{a-b}{(a+b)^2}$$

$$=1\times\frac{a-b}{(a+b)^2},\text{ or }\frac{a-b}{(a+b)^2}$$

*Ex.* (b). $\dfrac{a^2-ab}{3b^2}\times\dfrac{2ab}{a^2-b^2}\times\dfrac{ab+b^2}{a^2+2ab}$

$$\frac{a(a-b)\times 2ab\times b(a+b)}{3b^2\times(a+b)(a-b)\times a(a+2b)}$$

$$\frac{a\cdot b\cdot b(a-b)(a+b)}{a\cdot b^2(a-b)(a+b)}\times\frac{2a}{3(a+2b)}=\frac{2a}{3a+6b}$$

1. Find the prime factors of the numerators and denominators of the given fractions.

2. Indicate the multiplication in one fraction.

3. Form a fraction of the factors common to the numerator and denominator in the indicated multiplication. This fraction equals 1.

4. The remaining factors will form the product required.

1. $\dfrac{1-a}{b}\times\dfrac{ad}{1-a^2}$

2. $\dfrac{a^3-1}{a+1}\times\dfrac{37}{a^2+a+1}$

3. $\dfrac{a^4-b^4}{a-b}\times\dfrac{a^2-b^2}{a^2+b^2}\times\dfrac{am}{(a+b)^2}$

4. $\dfrac{a^2+ab+b^2}{a^3-b^3}\times\dfrac{(a-b)^2}{a^3+b^3}$

5. $\dfrac{x^4+a^4}{x^2+a^2}\times\dfrac{x^3-a^3}{x^2-a^2}\times\dfrac{x+a}{5}$

6. $\dfrac{m^2-mn+n^2}{21}\times\dfrac{3(m+n)^2}{m^3+n^3}$

7. $\dfrac{x^6 - a^6}{x^2 - ax + a^2} \times \dfrac{3x^2 - 3a^2}{(x^3 - a^3)(x + a)}$

8. $\dfrac{3(a^2 + b^2 + ab)}{a^2 + b^2 - ab} \times \dfrac{a^3 + b^3}{a^3 - b^3}$

9. $\dfrac{a^3b + 2a^2b^2 + ab^3}{3ab(a+b)^2} \times \dfrac{15a^3b^3}{ab}$

10. $\dfrac{(a + b)^2 - (c + d)^2}{(a + b) + (c + d)} \times \dfrac{1}{a^3 - c^3}$

11. $\dfrac{c^3 - 8}{17ab} \times \dfrac{ab^2 - b^3}{c^2 + 2c + 4} \times \dfrac{51a^2}{c - 2}$

12. $\dfrac{1 - 3y + 3y^2 - y^3}{1 - y^2} \times \dfrac{3 + 3y}{1 - 2y + y^2}$

13. $\dfrac{a^2m^2 - 9}{a^2m^2 - 6am} \times \dfrac{am + 3}{9 + 6am + a^2m^2}$

14. $\dfrac{a^2 - a - 6}{a^2 - 4} \times \dfrac{a^2 - a - 6}{a^2 - 6a + 9}$

15. $\dfrac{27 - x^3}{3(3 + x)} \times \dfrac{9 + 6x + x^2}{11(9 - x^2)}$

16. $\dfrac{x^2 - x - 6}{x^2 + 3x - 18} \times \dfrac{3ax + 18a}{x^2 + 5x - 14}$

17. $\dfrac{a^2(a^2 + a - 42)}{a^5(a^2 - 49)} \times \dfrac{a^3(a - 7)}{a^2 - 12a + 36}$

18. $\dfrac{37a^5c(a + b - c)^2}{5ay + 5by - 5cy} \times \dfrac{125acy}{111a^6c^2}$

19. $\dfrac{a^2b^2c^2 - 1}{a^2b^2c^2 - 9} \times \dfrac{a^2b^2c^2 + 2abc - 3}{a^2b^2c^2 - 2abc + 1}$

20. $\dfrac{8a^6 - b^3}{2a^3bc - ab^2c} \times \dfrac{121a^3b^3c^3}{4a^4 + 2a^2b + b^2}$

21. $\left(\dfrac{1}{a} - \dfrac{1}{b}\right) \times \left(a - \dfrac{a^2}{b}\right) \times \dfrac{ab^3c(b + a)}{b^2 - a^2}$

22. $\left(1 - \dfrac{2a}{1 + a}\right)\left(1 + \dfrac{2a}{1 - a}\right)\left(1 - \dfrac{2a}{1 - a}\right)\left(1 + \dfrac{2a}{1 + a}\right)$

23. $\left(\dfrac{1}{x} + \dfrac{1}{c} + \dfrac{1}{d}\right) \times \dfrac{x^2cd}{xd + xc + cd}$

24. $\left(\dfrac{2a}{2a - b} - 1\right)\left(\dfrac{8a^3}{b^3} - 1\right) \times \dfrac{4a^2 - b^2}{8a^3 - b^3}$

25. $\dfrac{a^2}{a^2 - ab + b^2} \times \left(a - \dfrac{ab^2 - b^3}{a^2 - b^2}\right)\left(a - \dfrac{ab - b^2}{a}\right)$

26. $\left(\dfrac{a^3 - b^3}{a - b} - 3ab\right) \times \dfrac{a^2 - b^2}{(a - b)^2}$

27. $b^2\left(a - \dfrac{b^2}{a}\right) \times \dfrac{3ab(a + b) + b^3 + a^3}{(a + b)^2 \times 2a} \times \dfrac{1}{a^2 - b^2}$

28. $\left(\dfrac{x}{2} + \dfrac{3}{7} + \dfrac{5c}{8}\right) \times \dfrac{56(x^2 - c^2)}{x^3 - c^3} \times \dfrac{x^2 + cx + c^2}{x + c}$

29. $\dfrac{ax}{x^2 - a^2} \times \dfrac{(a^2 - x^2)^3mn}{a^4 - 2a^2x^2 + x^4} \times \dfrac{1}{m^2n^2}$

30. $\left(a - 1 + \dfrac{2}{a + 1}\right) \times \dfrac{5a + 5}{2a^2 + 2} \times \left(a + b - \dfrac{a^2 + b^2}{a - b}\right)$

31. $\left(x - 3 - \dfrac{3x(3 - x)}{x - 2}\right) \times \left(1 - \dfrac{3}{x + 1}\right) \times 3\tfrac{4}{7}$

32. $\left(\dfrac{1}{x + y} + \dfrac{1}{x - y}\right) \times \left(\dfrac{x^2}{3} - \dfrac{y^2}{3}\right) \times \left(\dfrac{x}{y} - \dfrac{y}{x}\right) \times \dfrac{3}{x^2 - y^2}$

**EXERCISE 42**

**44.** Perform the division indicated:

Given, $\frac{a}{b} \div \frac{c}{d}$

Dividing, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b \times \frac{c}{d}}$ § 39, (*b*) 3

Simplifying, $\frac{a}{b \times \frac{c}{d}} \times \frac{d}{d} = \frac{ad}{bc}$ § 39, (*b*) 5

But $\frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}$

In the result the divisor reappears inverted. Hence, in dividing one fraction by another fraction, for convenience, multiply the dividend by the inverted divisor.

1. $\frac{a-2}{a-3} \div \frac{a-4}{a-3}$

2. $\frac{x^3-n^3}{x^3+n^3} \div \frac{(x-n)^2}{x^2-n^2}$

3. $\frac{21a^2-3a}{6a^3+12a^2} \div \frac{2a-1}{a^2+2a}$

4. $\frac{16a^2-9b^2}{a^2-4} \div \frac{4a-3b}{a-2}$

5. $\frac{a^2x^2+3ax}{4a^2-1} \div \frac{ax+3}{2a+1}$

6. $\frac{a^2-a-20}{a^2-25} \div \frac{a+1}{a^2+5a}$

7. $\frac{a^4-x^4}{(a^2-x^2)^2} \div \frac{(a^2+x^2)^3}{a^2+x^2}$

8. $\frac{x^2+7x-8}{2a+x} \div \frac{x^2+x-56}{ax^2+2a^2x}$

9. $\frac{a^2+7a+12}{a^2+2a-15} \div \frac{a+4}{a+5}$

10. $\frac{c^2-6c+8}{c^2+2c+1} \div \frac{c-4}{c+1}$

11. $\frac{x^2-2x+1}{a^2+4a+4} \div \frac{x-1}{a+2}$

12. $\frac{c^2-a^2}{c^3-1} \div \frac{3(c-a)}{1-c}$

13. $\frac{x^3b+2x^2b^2+xb^3}{x^4-b^4} \div \frac{xb}{x-b}$

14. $\frac{a^2+2a-3}{a^2-2a-3} \div \frac{a^3-1}{a^3+1}$

15. $\frac{n^2+no+o^2}{n^3-o^3} \div \frac{211}{(n-o)^2}$

16. $\frac{a^3-a^4+a^5}{1-(a-a^2)} \div \frac{a^4-2a^3}{a^2-2(a-2)}$

**EXERCISE 43**

**45.** Simplify the complex fractions:

1. $\dfrac{a+\dfrac{b}{x}}{a-\dfrac{b}{c}} \times \dfrac{cx}{cx} = ?$

2. $\dfrac{x-3+\dfrac{2}{x}}{x+1-\dfrac{6}{x}} \times \dfrac{x}{x} = ?$

3. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{b}{a}-\dfrac{a}{b}} \times \dfrac{ab}{ab} = ?$

4. $\dfrac{\dfrac{a}{b}-\dfrac{b}{c}-\dfrac{c}{a}}{\dfrac{1}{c}-\dfrac{1}{a}-\dfrac{1}{b}} \times \dfrac{abc}{abc} = ?$

5. $\dfrac{a+\dfrac{3a}{a-4}}{a-\dfrac{3a}{a-4}} \times \dfrac{a-4}{a-4} = ?$

6. $\dfrac{1+\dfrac{b}{a-b}}{1-\dfrac{b}{a+b}} \times \dfrac{a^2-b^2}{a^2-b^2} = ?$

7. $\dfrac{\dfrac{1}{1-x}+\dfrac{1}{1+x}}{\dfrac{1}{1-x}-\dfrac{1}{1+x}} \times \dfrac{1-x^2}{1-x^2} = ?$

8. $\dfrac{x^2-x-6}{\dfrac{4}{x^2}-1} \times \dfrac{x^2}{x^2} = ?$

9. $\dfrac{\dfrac{a}{2}-\dfrac{n}{6}}{\dfrac{x}{3}-\dfrac{n}{2}}$

10. $\dfrac{c-\dfrac{1}{c}}{\dfrac{1}{c}+1}$

11. $\dfrac{\dfrac{a^2}{b^2}+1}{\dfrac{b^2}{a}-1}$

12. $\dfrac{\dfrac{3x}{5z}+\dfrac{4}{z}}{z+\dfrac{2x}{3}}$

13. $\dfrac{\dfrac{9}{a}-a}{\dfrac{2}{a^2}-\dfrac{1}{a^3}}$

14. $\dfrac{\dfrac{b}{c^2}+\dfrac{c}{b^2}}{\dfrac{1}{b^2}+\dfrac{1}{c^2}}$

15. $\dfrac{\dfrac{n}{m}+\dfrac{m}{n}}{\dfrac{1}{m^2}-\dfrac{1}{n^2}}$

16. $\dfrac{\dfrac{a^2+c^2}{a+b}}{\dfrac{1}{a^2-b^2}}$

## TEST O

1. Define: fraction; proper fraction; complex fraction; integral number; mixed number.

2. Write a rule for the process required in each exercise 36–43.

3. Is $x^2 + x$ integral in form? If $x = \frac{1}{5}$, is it integral in value? If $x = 5$?

4. Why is not the value of a fraction reduced when the fraction is reduced to its lowest terms?

5. Why is not the value of a fraction increased when the fraction is changed to another fraction of higher denomination?

6. Is the value of a mixed number changed when the mixed number is changed to a fraction?

7. Explain why the value of a complex fraction is not changed when the complex fraction is changed to a simple fraction.

8. Answer question 9, § 39, Test M.

9. Dividing the numerator of a fraction, § 39 (*b*) 4, as well as multiplying its denominator, § 39 (*b*) 3, divides the fraction. Why is § 39 (*b*) 3 alone referred to in § 44, Exercise 42?

10. Which of the grouping marks is the line between the numerator and the denominator of a fraction?

11. $$\frac{x^3 + 4x^2 - 5}{x^2 + x - 2} = x + 3 + \frac{-x + 1}{x^2 + x - 2}.$$

Write this mixed number, giving the minus sign to the fraction.

12. Are $x + 3 - \dfrac{-x + 1}{-x^2 - x + 2}$ and $x + 3 - \dfrac{1 - x}{2 - x - x^2}$ correct forms? Is there any objection to either?

13. (*a*) $+\dfrac{3x-4}{x^2+2x-1}$  (*b*) $-\dfrac{3x-4}{-(x^2+2x-1)}$

(*c*) $-\dfrac{3x-4}{-x^2-2x+1}.$

In changing (*a*) to (*b*) how many fraction signs are changed? In changing (*a*) to (*c*) how many fraction signs and how many term signs are changed?

**Note.** In Exercises 13–16, the signs of the fraction, the numerator as a whole, and the denominator as a whole, are for brevity called *fraction signs.*

14. What is the limit to the number of term signs that may be changed by changing one fraction sign?

15. (*a*) $+\dfrac{1}{(b-c)(b-a)}.$  (*b*) $-\dfrac{1}{-[(b-c)(b-a)]}.$

(*c*) $-\dfrac{1}{(b-c)\times-(b-a)}.$  (*d*) $-\dfrac{1}{(b-c)(a-b)}.$

How many fraction signs are changed in transforming (*a*) into (*b*)? In transforming (*a*) into (*c*) how many factor signs are changed? How many term signs are changed in changing one factor sign in (*d*)? What is the limit?

16. (*a*) $+\dfrac{1}{(c-a)(c-d)}.$  (*b*) $+\dfrac{1}{-(c-a)\times-(c-d)}.$

The change of how many factor signs will not change a fraction sign? Changing the sign of an intern changes the signs of how many terms?

17. Given $-\dfrac{abc}{(a-b)(b-c)(c-a)}.$ Change the sign of one factor of the denominator; of two factors; of three factors. Which of these changes will change the sign of the denominator as a whole?

**18.** Write a review of sign changes in fractions: (1) The signs of a fraction independent of the term signs of the numbers composing it; (2) The possible changes illustrated and explained; (3) Possible changes in signs of factors without change of product; hence (4) Changes in numerator or in denominator; (5) The forming of a negative fraction from a remainder and a divisor.

**19.** $\frac{m+1}{m^2+3m}$, containing powers of $m$ only, is a proper fraction because the numerator is of a lower degree than the denominator. Substitute $\frac{1}{3}$ for $m$, reduce to simplest form, and show that a proper literal fraction may be greater than 1. Can $m$ denote a negative number?

EXERCISE 44

**46.** Raise the given fractions to the power indicated:

Example (*a*). $\left(-\frac{a}{b}\right)^2 = -\frac{a}{b} \times -\frac{a}{b} = +\frac{a^2}{b^2}$

Ex. (*b*). $\left(-\frac{1}{c}\right)^3 = -\frac{1}{c} \times -\frac{1}{c} \times -\frac{1}{c} = -\frac{1}{c^3}$

Ex. (*c*). $\left(-\frac{3x^2}{2y^2z}\right)^3 = -\frac{3 \cdot 3 \cdot 3 \,.\, x^2 \cdot x^2 \cdot x^2}{2 \cdot 2 \cdot 2 \cdot y^2 \cdot y^2 \cdot y^2 \cdot z \cdot z \cdot z}$

Ex. (*d*). $\left(-\frac{3x^2}{2y^2z}\right)^3 = -\frac{3^3 \cdot (x^2)^3}{2^3 \cdot (y^2)^3 \cdot z^3} = -\frac{27x^6}{8y^6z^3}$

Ex. (*e*). $\left(-\frac{x^2-y^2}{x^3}\right)^4 = +\frac{(x^2-y^2)^4}{x^{12}}$

and $\left(+\frac{m-x}{a+d}\right)^2 = \frac{m^2-2mx+x^2}{a^2+2ad+d^2}$

The fraction with its proper sign is used as a factor as many times as there are units in the index of the required power. To shorten the process the exponent of each letter may be multiplied by the index, as in example (*d*), when the terms of the fraction are monomials. When the numerator or denominator is a polynomial, the multiplication must be indicated or performed as in (*e*).

1. $\left(+\frac{2a^3b}{5cd}\right)^2$

2. $\left(-\frac{3mn}{4ab}\right)^2$

3. $\left(-\frac{11a^5x}{7y}\right)^2$

4. $\left(-\frac{9c^4d}{ab^2}\right)^3$

5. $\left(+\frac{-m^2n^3}{-c^3d}\right)^5$

6. $\left(-\frac{a-b}{c+d}\right)^2$

7. $\left(-\frac{m^2-n^2}{mx}\right)^3$

8. $\left(+\frac{(a+b)(a-b)}{3c(a^2+b^2)}\right)^2$

9. $\left(+\frac{-(a-c)(c^2+ac+a^2)}{-(c^2-a^2)}\right)^2$

10. $\left(+\frac{(a^4+b^4)(a^2+b^2)(a+b)(a-b)}{a^4-b^4}\right)^3$

### EXERCISE 45

Example (*a*). $\left(\frac{1}{x}-\frac{1}{y}\right)^2=\left(\frac{1}{x}\cdot\frac{1}{x}\right)-2\left(\frac{1}{x}\cdot\frac{1}{y}\right)+\left(\frac{1}{y}\cdot\frac{1}{y}\right)$

$$=\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}$$

Ex. (*b*). $\left(4+\frac{a}{b}\right)^2=(4\cdot 4)+2\left(4\cdot\frac{a}{b}\right)+\left(\frac{a}{b}\cdot\frac{a}{b}\right)$

$$=16+\frac{8a}{b}+\frac{a^2}{b^2}$$

To find the root of a trinomial square, inquire: Of what is the first term a square? Of what is the third term a square? Does the second term show that the binomial root is a sum or a difference?

Perform the operations indicated:

1. Square $\frac{a}{b} - \frac{b}{a}$.

2. Find the square root of $\frac{x^2}{a^2} - 2 + \frac{a^2}{x^2}$.

3. Square $\frac{1}{b} - \frac{1}{d}$.

4. Find the square root of $\frac{1}{m^2} - \frac{2}{mn} + \frac{1}{n^2}$.

5. Square $\frac{2}{c} + \frac{x}{d}$.

6. Find the square root of $\frac{9}{a^2} + \frac{12}{ab} + \frac{4}{b^2}$.

7. Square $\frac{a}{x} + 5$.

8. Find the square root of $121 - \frac{22x}{c} + \frac{x^2}{c^2}$.

9. Square $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

10. Cube $\frac{1}{x} - \frac{1}{y}$.

11. Raise to the fourth power $\frac{1}{a} - \frac{1}{b}$.

Determine by inspection the roots of the following powers:

12. $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{2}{xy} + \frac{2}{xz} + \frac{2}{yz}$.

13. $\frac{1}{b^3} - \frac{3}{b^2d} + \frac{3}{bd^2} - \frac{1}{d^3}$.

14. $\frac{1}{x^4} - \frac{4}{x^3y} + \frac{6}{x^2y^2} - \frac{4}{xy^3} + \frac{1}{y^4}$.

*Example.* Expand $(2x - a^2)^3$

$$(2x)^3 - 3(2x)^2(a^2) + 3(2x)(a^2)^2 - (a^2)^3$$

$$8x^3 - 12x^2a^2 + 6xa^4 - a^6$$

**15.** Expand $(2a + 7)^3$.

**16.** Expand $(2ac - x)^3$.

**17.** Expand $(a^2x - 2c)^3$.

**18.** Expand $(2x - a^2)^4$.

**19.** Expand $(a^2 - 3b)^4$.

**20.** Expand $(1 + 2x)^4$.

**21.** Expand $(x^2 - y^2)^4$.

**22.** Expand $\left(\frac{2x}{3} - \frac{3y}{4}\right)^3$.

**23.** Expand $\left(\frac{x}{2} + 1\right)^3$.

**24.** Expand $\left(\frac{a}{3} - \frac{x}{2}\right)^4$.

## VIII. Miscellaneous Exercises and Problems

**47.** 1. Multiply $(a+b)(x+y)$; $(a+b)(x-y)$; $(a-b)(x+y)$; $(a-b)(x-y)$; and find a method of factoring such quadrinomials as result.

2. What is the difference between $(a+b)^2(a-b)^2$ and $[(a+b)(a-b)]^2$?

3. Square $a-x-y$ without multiplying. In the same way square $a-2x-3y$; $2x^3-x+5$; $a+x+y+z$.

4. Simplify each fraction:

$$\frac{1}{\left(1+\frac{z}{x}\right)\left(1-\frac{z}{x}\right)};\quad \frac{1}{\left(1-\frac{x}{y}\right)\left(1-\frac{x}{y}\right)};\quad \frac{1}{\left(1-\frac{x}{z}\right)\left(1-\frac{y}{z}\right)}.$$

5. What number must be added to each of the following binomials to make of each a trinomial square:

$$x^2-2xy;\ 4+4x;\ 9-6y;\ y^2+6y;\ x^2+\frac{b}{a}x;\ x^2-\frac{3}{4}x?$$

6. Find the continued product, grouping the factors so that the multiplication may be done by inspection:

$$(x-y)(x^2-xy+y^2)(x^2-y^2)(x^3+y^3)(x^2+xy+y^2)(x^2+xy+y^2).$$

7. Why should the denominators of all fractions to be added or subtracted be arranged according to the ascending or the descending powers of the same letter?

8. All powers of negative and of positive monomials are positive with one exception. What is the exception?

**9.** In the number $1a^1+1$, how many of the ones may be omitted? In $\frac{3+3a^2b}{6a} \div \frac{3}{3} = \frac{1+1a^2b}{2a}$?

**10.** Reduce $\frac{6x^3-4x^2+2x-4}{2x^3+x^2-x-2}$ to a mixed number.

**11.** Reduce $\frac{a^4+2a^2+1}{a^2+a+1}-(a-a^2-1)$ to an improper fraction.

**12.** Unite and simplify $\frac{2}{n^2-n-2}-\frac{1}{n^2-1}+\frac{2}{n^2-3n+2}$.

**13.** Add the first and second terms, their sum and the third, and so on: $\frac{1}{a-x}-\frac{1}{a+x}-\frac{2x}{a^2+x^2}-\frac{4x^3}{a^4+x^4}$.

**14.** What number must be added to $a^2+2ab$ to make $(a+b)^2$? To $x^2-2xy+y^2$ to make $(x+y)^2$?

**15.** Subtract a number from $(a+b)^2$, and make $(a+b+3)(a+b-3)$. From $(x-y)^2$, and make $(x-y+5)(x-y-5)$.

**16.** Divide $2x$ by $1+3x^2$. Give four terms and a fractional term in the quotient.

**17.** The powers of which numbers are less than the base? $(.5)^2$; $(\frac{1}{5})^3$; $(2.1)^3$; $(ab^2)^3$; $\left(\frac{1}{ab^2}\right)^3$; $(3\frac{1}{3})^2$; $(.004)^3$; $(1)^5$; $(a^0)^5$; $(a^{-2})^5$; $\left(\frac{1}{a^2}\right)^5$.

**18.** Separate into three fractions, and reduce each to its lowest terms: $\frac{5x^2y-10xy^2+15y^3}{10x^2y^2}$.

**19.** Simplify $5a-(2b-c)-(5a+3b+\overline{4c-2d})$, and indicate the subtraction of the result from $5c-4d$.

**20.** Subtract $\frac{a}{b}$ from $\frac{a^2}{b^2}$, add $\frac{a}{3b}$ to the remainder, and give the reciprocal of the sum.

21. Subtract $a$ from 0, $c$ from 1; to the product of the differences add $2a - ac$, and multiply the result by its reciprocal.

22. Simplify $5\{1 - 3[1 - 2(1 - 3m)]\}$ and find what must be added to the result to make $15 - 79m$.

23. By how much do $5x^3 - x^2 + 1$ dollars, $x^2 + 5$ dimes, and $x^3 + 11x^2 - 3$ cents fall short of paying a bill of $6x^3 + x^2 + 2$ dollars?

24. Given, $17(5x + 3a)^2 - 2(40x + 27a)(5x + 3a) = 25x^2 - 9a^2$. Divide both members by a common factor.

25. A man bought $7ca$ spades at $2a$ dimes each; 5 picks at $3ca^2$ dimes each; $2c$ rakes at $5a^2$ cents each. How many dollars did he pay?

26. Is $3ab(a - b)^2 + (a - b)^4$ equal to $(a - b)(a^3 - b^3)$?

27. What is the difference between the opposite of a number and the reciprocal of a number? If a number is negative, what is the quality of its opposite? Of its reciprocal?

28. Does any axiom give authority for changing all the signs in an equation? For inverting a fraction in division?

29. Is "Let $x =$ the number" an equation or a sentence? For what word is the sign $=$ used? What word is used for the sign $=$ in reading $x = 5$?

30. Divide $a$ by $a$ as in arithmetic, and again as in algebra. Are the quotients equal or unequal? What axiom applies here?

31. In the product of $a^2 + 2ab + b^2$ by $a + b$, in what term may $a^0$ be written? $b^0$?

32. May $a^0$ as a factor be omitted without loss of value? May $a^0$ as a term be omitted?

**33.** Reduce $\frac{r^3-1}{1+r}$ to a mixed number, reduce $\frac{r^4+r^2+1}{r^2-r+1}$ to its lowest terms, and add the results.

**34.** Simplify $\frac{a+b}{m} \div \frac{b^2-a^2}{m^2}\left(\frac{1}{x} \div \frac{m}{a-b}\right)$.

**35.** Divide 1 by $1-x$. How many terms in the quotient? Is there a visible law in its formation?

**36.** If $A = m^4+m^2n^2+n^4$, $B = m^4-m^2n^2+n^4$, and $C = m^4-n^4$, what is the value of $ABC + \frac{A+B}{2} - C$?

**37.** Square $xy+yz+zx$, $x \pm y$, and $a \pm 5$.

**38.** Reduce $\frac{x^2 \pm 2xy + y^2}{x^2-y^2}$ to its lowest terms.

**39.** Square 37 expressed as $(40-3)$; 41, as $(40+1)$; 99, as $(100-1)$.

**40.** Simplify $\frac{2}{\frac{1}{x}-\frac{1}{y}} + \frac{y}{1-\frac{y}{x}} - \frac{x}{1-\frac{x}{y}}$.

**41.** Simplify $\frac{\frac{a^2+b^2}{b}}{\frac{1}{b}-\frac{1}{a}} \times \frac{a^2-b^2}{a^3+a^2b}$.

**42.** Show that $(m+n)(m+n-1) = m(m-1) + 2mn + n(n-1)$.

**43.** If the value of $x$ obtained by solving a problem will satisfy the equation formed, but will not satisfy the conditions of the problem, where is the error?

**44.** Given, $ax - b(x-1) - c = 0$. Find the value of $x$, substitute it for $x$, and prove the equation.

45. Multiply $a^3+b^3$ by $a-b$ and divide by $a+b$; or, divide $a^3+b^3$ by $a+b$ and multiply by $a-b$. Is there an advantage in either order?

46. Given, $(x^3+x^2y+xy^2+y^3)\div(x+y)$. Group the terms so that the division may be done by inspection.

47. By inspection, divide: $(a+b)^3+c^3$ by $a+b+c$; $(a+b)^3-c^3$ by $a+b-c$; $(a+b)^4-c^4$ by $(a+b)^2-c^2$.

48. Expand $[a^2-(a-b)x-ab][a^2+(a+b)x+ab]$, and express the product in two factors.

49. Given,

$$a^2(b-c)+b^2(c-a)+c^2(a-b)+(b-c)(c-a)(a-b)=0.$$

Is the equation true?

50. What must be added to $(x+y+z)^2$ that the result may be $(x-y-z)^2$?

51. By inspection, divide both members of the given equation, $11\,a^3b(a+b)^3=a^3+b^3+3\,ab(a+b)$, by $(a+b)$.

52. If $(a+4)(a+3)=a^2+7\,a+c$, what is the numerical value of $c$?

53. Arrange $-a+b+c+d$ and $a-b+c+d$ as a sum and a difference, and obtain their product by inspection.

54. Can $4(ab+cd)$ be obtained by simplifying

$$(a+b+c-d)(a+b-c+d)+(a-b+c+d)(-a+b+c+d)?$$

55. The higher powers of what numbers are less than their base? What powers of numbers and of their opposites are the same?

56. Factor: $9+x^2-6\,x$; $x+x^2-6$; $-b^3+a^3$.

57. Factor: $2\,bc+c^2+b^2-2\,ab+a^2-2\,ac$.

58. Factor: $5(a^3-b^3)+10\cdot\overline{a-b}+c(a^2-2\,ab+b^2)$.

59. Factor: $\dfrac{x^2}{y^2z^2}-\dfrac{4\,x}{3\,yz}+\dfrac{4}{9}$; $\dfrac{p^2}{q^2}+2+\dfrac{q^2}{p^2}$.

60. Factor: $ax+ay+bx+by$; $cm+cn-dm-dn$; $cm+2c-3m-6$; $a^2-5a-3a+15$.

61.

$$\frac{(x^3-y^3)(x^2+xy+y^2)(x^2-y^2)(x-y)(x^2+y^2)(x^2-xy+y^2)(x-y)(x+y)}{(x^6-y^6)(x^4-y^4)(x-y)^2}.$$

Determine by inspection the simplest form of the given fraction.

62. What is cancellation?

63. Express the ratio $\frac{ac}{dx}$ in another form, with terms $m$ times as large; with terms $\frac{1}{2}$ as large; with terms $\frac{1}{n}$ as large. Does the relation of $ac$ to $dx$ remain the same?

64. What is the common difference in a series of consecutive even numbers? In a series of consecutive odd numbers? Represent a series of five even numbers, $n$ being the middle number, and a series of five odd numbers, $p$ being the middle number.

65. What are the prime factors of the coefficient of the fifth term of $(M-R)^{10}$?

66. Reduce $\frac{ac+bd+ad+bc}{af+2bx+2ax+bf}$ to its lowest terms.

67. Divide $a$ into two parts, such that the second part shall equal $m$ times the first part plus $n$.

68. Form the proportion required in the statement: "The product of two numbers is 96; the difference of their cubes is to the cube of their difference as nineteen is to one."

69. Find by inspection the H. C. F. and L. C. M. of the numbers in each of the three following expressions: $(2x-4)(3x-6)$; $(x^2-9)(2x-6)$; $(4x-8)(5x-10)$.

70. Find the difference of the squares of the two odd numbers $2a+1$ and $2b+1$; also the difference of the

squares of $1-4c$ and $1-4d$; of $(11)^2$ and $9^2$. What one of the digits is a common factor of the differences?

71. Given, $(a^2-b^2)^2=(a^3+b^3)-ab$. Factor the groups; substitute 1 for $a+b$. What identity results?

72. If $7(x-y)=3(x+y)$, which is the greater, $x$ or $y$?

73. Compare the coefficient of the middle term in the expansion of $(p-q)^4$ with the sum of the coefficients of the two middle terms in the expansion $(R-B)^3$.

74. Express the product of $x^2+6-7x$, $7x-18+x^2$, $x^3-1$ in their prime factors.

75. "The sum of any two consecutive whole numbers is equal to the difference of their squares." Let $x$, $x+1$, $x+2$, $x+3$ be four consecutive whole numbers, and test the statement by three trials.

76. Given $+10a^2b^3$, a term in a power of a binomial. Write all the terms of the power.

77. The sum of three consecutive whole numbers exceeds the greatest of them by 19. What are the numbers?

78. If the sum of two numbers is 20, and their difference is 6, what are the numbers? If the sum of two numbers is $m$, and their difference is $n$, what are the numbers?

79. $x^3+3x^2y+3xy^2+y^3$ is a *homogeneous* expression. Of what degree is each term? Are the terms of every power of $x+y$ of the same degree?

80. Define: (1) algebraic notation; (2) formula; (3) symbols of number; (4) symbols of operation; (5) principle; (6) conditions of a problem; (7) solution of a problem; (8) solution of an equation.

81. What is a science? What is mathematics? What is Arithmetic? What is Algebra?

## PROBLEMS LEADING TO SIMPLE EQUATIONS

**48.** 1. A gardener planted 25 dahlias in 5 rows, each row having 2 less plants than the preceding row. How many plants were there in the third row?

2. In a bookcase there are 117 books. How many volumes of the encyclopædia are on one shelf if 11 times one-half the number equals the remainder of the books?

3. How many canoes were there on the river if the product of 7 and the number, less the sum of the number and 7, is five times the number?

4. How many Japanese lanterns were lighted if the sum of one-fourth and one-fifth of the number is 5 less than one-half of the number?

5. How many passengers are there inside the coach if three times the number less 9 equals the number on top, and there are in all 16 persons, including the driver?

6. What is the date on the Boston Stone if one-ninth of the number of years added to 193 equals one-third of the number of years less 193?

7. Benjamin Franklin lived from 1706 to 1790. How old was he when he entered the Boston Latin School if nine times his age at that time added to 12 years equalled his age when he died?

8. In what year is it probable that La Salle was at the site of Chicago? If to the number of years two-thirds of the number is added, the sum will be 557 less than twice the number.

9. A man bought a boat which he repaired and painted, at an expense of $180. He paid for repairs and painting one-half as much as the boat cost. What did the boat cost?

10. Some boys camped in the Maine woods. A few boys had an accident; four were ill. Those who had no accident and were not ill were two-thirds of the whole number, and the whole number was nine times the number that had an accident. How many boys were there?

11. Three fishermen went out for fish. The second caught five-thirds as many as the first; the third caught one and one-eleventh times as many as the first, who said his number was the same as the number of years in his age. The number they all caught was eight less than four times the first's years. How many did each catch?

12. Find two consecutive numbers such that the fourth part plus the fifth part of the less equals the third part plus the ninth part of the greater.

13. A fisherman caught sixty-four lobsters. He found some lobsters less than ten inches long, and he threw them back into the sea. He kept twice as many as he threw back. He gave away one-third as many as he kept, and sold thirty-one. How many did he give away?

14. A man left his property to be divided among his three children. The share of the eldest was twice that of the second, and the share of the second was twice that of the youngest. The eldest received $2100 more than the youngest. How much did each receive?

15. A man can walk a certain distance in four hours. If he were to increase his rate by one-fifteenth, he could walk one mile more in that time. What is his usual rate?

16. A little girl who sold flowers traded with one who sold oranges. Of roses at the rate of two for seven cents, and pinks at the rate of four for five cents, she gave the same number. She received two-thirds as many oranges at three cents apiece as she gave flowers, and the orange girl

owed her two and one-fourth cents. How many oranges did she get? If she had given two less pinks, how much would she owe the orange girl?

**17.** If the sum is 20 and the difference 6, what are the numbers? If the sum is $m$ and the difference $n$, what are the numbers?

**18.** There was food in camp for a certain number of men for 6 days. Five more men came and the food lasted but 4 days. How many men at first? Suppose there was food for the men for $p$ days; that $r$ more men came; that the food lasted only $q$ days. How many men later?

**19.** A bootblack earned $9.65 in one week. The number of men and of boys who had a "5-cent shine" were as 23 : 11. The number of men who had russet shoes dressed at 10 cents a pair was two times the number of boys who had a "shine," and the number of girls who brought him their russet shoes was five less than the boys who had a "shine." One girl gave a quarter, and did not take change. How many girls patronized the bootblack?

**20.** On the lower story of a building there are 24 panes of glass in each window; on the next story there are 20 panes; on the next, 16 panes. The ratio of the number of windows of the three kinds is 4 : 2 : 1. The whole number of panes lacks 32 of being nine times the L. C. M. of the sum of the panes in three windows, one of each kind. How many windows are there?

**21.** A newsboy received a number of papers on Thursday, of which he sold one-half. He received 5 less on Friday, and sold one-third of them. On Saturday he received 5 more than he received on Thursday, and sold three-fifths of them. He sold thirty papers. How many did he sell on Saturday?

**22.** A boy slid down hill in a certain number of minutes. It took him seven-thirds as long to walk up, and it took seven-ninths of an hour to make eleven round trips. How many minutes were required to slide down?

**23.** Two men started at the same time, from the same place, and travelled in opposite directions. One travelled at the rate of 20 miles a day, the other at the rate of 30 miles a day. Their roads met after they had made a circuit of 300 miles. How many days did they travel?

**24.** Together two persons, E and F, earned $3000 a year. At the end of five years they had saved $6000. E spent each year $200 more than F spent. What did each spend annually?

**25.** A girl tried at home 35 recipes taught at the cooking-school. One-half as many compounds were burned as were underdone, and four-thirds as many as the number burned were poor because she had forgotten directions; and one was lost because she spilled the milk required for it. Her successes were three-fifths of the number of recipes. What number were poor?

**26.** A history has a certain number of maps, 3 more than one-half as many illustrations as maps, one-fourth as many portraits as maps, and as many autograph letters as portraits. These occur separately on 59 pages. How many maps are there?

**27.** The average price of each reading book in a series of five books is 45 cents. The difference in price between two consecutive readers is one dime. What is the price of the fifth-grade reader?

**28.** If one number is subtracted from another number the difference is 7; if the second is divided by the first the quotient is 7. What are the numbers? Verify the result.

29. The length of the road is 5000 units. The perimeter of the cart wheel is $\frac{1}{8}$ plus a certain number of the same units. A teamster ordinarily requires 64 minutes for the distance, but for a very heavy load he requires one hour and one-sixth. The longer time allows one second more for the revolution of the wheel. What is the circumference of the cart wheel?

30. A boy can shovel the snow from a sidewalk in $b$ hours; and his brother, in $c$ hours. In what time can both together shovel it?

31. A shepherd boy said that if he had 35 less sheep to watch, he should have three fifths of one-half as many as he has now. How many has he?

32. The difference between the numerator and the denominator of a fraction is 4. If 7 is added to each term, the fraction equals $\frac{5}{6}$. What is the fraction?

33. If a fraction is reduced to its lowest terms, it is $\frac{1}{m}$. If $p$ is added to the numerator and $q$ to the denominator, a fraction will be formed whose value is $\frac{1}{n}$. What is the fraction?

34. A cistern is one-third full. If 31 gallons of water are poured in, it will be half full. How many gallons will the cistern hold?

35. The perimeter of a square is 44 feet more than the perimeter of another square. The former has 187 square feet in area more than the latter. Find the length of a side of each square.

36. The circumference of a wheel is $m$ feet; of a smaller wheel, $n$ feet. In going a certain distance the smaller wheel makes $r$ revolutions more than the larger makes. What is the distance in feet? In rods?

37. The fore wheel of a carriage makes the same number of revolutions in going 400 yards that the hind wheel makes in going 475 yards. The circumference of the hind wheel is 3 feet greater than the circumference of the fore wheel. What is the circumference of each wheel?

38. Eighteen coins, half-dollars and quarter-dollars, amount to $6.50. How many coins of each kind?

## EXERCISES IN THE CONSTRUCTION OF PROBLEMS. II

**49.** In constructing problems make short, clear sentences; ask a definite question; solve the problems when they are complete; verify the equations; verify the statements of the problems.

The statements of a problem must be reasonable. If the correct solution of the equation leads to the conclusion that five and one-half men dug a ditch, or that a boy is older than his father, the problem should be reconstructed.

Construct problems from the suggestions which follow. If insufficient facts are called for in any one of the groups of questions, further inquiries should be made. If any statement is superfluous, it may be omitted. Answer the questions with consistent numerical numbers; be sure that an equality can be formed; decide which numbers shall be known and which unknown; write the problem.

1. What did your pocketbook cost? How many dollars and how many dimes are in it now? How many times the cost of the pocketbook have you spent? How much was the pocketbook with its contents worth before you went shopping?

2. How many fishing sloops can you see from the pier? How many yachts? How many more dories than yachts?

How many times as many sloops as yachts equals the product of the sum and the difference of the yachts and the dories?

3. What was the ratio of the number of boys who voted for a race to the number who voted for baseball? How many boys changed their vote so that the terms of the ratio were reversed?

4. What was the ratio of the number of spearmen to horsemen to light-armed archers? How many men in the battle?

5. What is the ratio of the number of words in the first stanza of "Hail, Columbia!" to the number of words in the first stanza of "America"? How many stanzas in each poem?

6. There are 268 words in Lincoln's Gettysburg Address: 196 are of one syllable, 46 are of two syllables, 26 are of three or four syllables.

7. How many bushels of sweet potatoes did Uncle Tom raise? How many bushels of peanuts? How many cents did he ask for a gourd? He sold five gourds, and set the prices of his potatoes and peanuts at a certain number of times what he received for the gourds. He sold enough bushels of potatoes to bring him the same sum that his peanuts and gourds brought, and a number of dimes besides.

8. At what rate did a boy buy apples? At what rate did he sell one-half of them? At what rate did he sell the other half? How many cents did he gain? How many apples did he buy?

Materials are always at hand; but the work of constructing problems of a kind easily solved need not be long continued.

# IX. Problems Leading to Simultaneous Equations

**50.** *Example A.* If the first of two numbers is multiplied by 7, and the second by 2, the sum of the products is 47. If 4 times the second is subtracted from 5 times the first, the remainder is 1. Find the numbers.

Let $x =$ the first number,

$y =$ the second number.

$$\left.\begin{aligned} 7x + 2y &= 47 \\ 5x - 4y &= 1 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

$$14x + 4y = 94 \qquad (3) \quad \text{Equation } (1) \times 2.$$

$$5x - 4y = 1 \qquad (2)$$

Add Eq. (3) and Eq. (2)

$$19x = 95 \qquad (4)$$

$$x = 5 \qquad (5)$$

$$35 + 2y = 47 \qquad (6) \quad \text{Value of } x \text{ substituted in } (1).$$

$$2y = 12 \qquad (7)$$

$$y = 6 \qquad (8)$$

Verification:

$$7 \cdot 5 + 2 \cdot 6 = 47 \quad \text{Values of } x \text{ and } y \text{ substituted in}$$

$$5 \cdot 5 - 4 \cdot 6 = 1 \quad \text{Equations (1) and (2).}$$

$$47 = 47$$

$$1 = 1$$

*Example B.* There are two numbers: 35 times the first plus 17 times the second equals 86; 56 times the first minus 13 times the second equals 17. What are the numbers?

Let $x =$ the first number,

$y =$ the second number.

| | | |
|---|---|---|
| $35x + 17y = 86$ | (1) | $\left[\begin{matrix}35 = 5 \times 7\\56 = 8 \times 7\end{matrix}\right]$ |
| $56x - 13y = 17$ | (2) | |
| $280x + 136y = 688$ | (3) | Eq. (1) $\times$ 8. |
| $280x - 65y = 85$ | (4) | Eq. (2) $\times$ 5. |
| $201y = 603$ | (5) | Dif. of (3) and (4). |
| $y = 3$ | (6) | Substitute in Eq. (1). |
| $35x + 51 = 86$ | (7) | |
| $35x = 35$ | (8) | |
| $x = 1$ | (9) | |

Verification: $35 \cdot 1 + 17 \cdot 3 = 86$ $\quad 86 = 86$

$56 \cdot 1 - 13 \cdot 3 = 17$ $\quad 17 = 17$

In each of the examples (*a*) and (*b*) there are two conditions, each condition leading to an equation. From the algebraic statement, $7x + 2y = 47$, an expression for $x$ may be obtained, $x = \frac{47 - 2y}{7}$; but the expression is not a value of $x$, since the expression contains the unknown $y$. In the same way, $y = \frac{47 - 7x}{2}$, and $y$ is also indeterminate, its value depending upon the value of $x$.

## TEST P

1. Find expressions for $x$ and $y$ from the algebraic statement of the second given condition in example (*b*). Can the value of either of two unknown numbers be obtained from one equation? Reasons?

2. In the solution of example ($A$), why are the terms of equation (1) multiplied by 2 to make equation (3)?

3. Why are equations (2) and (3) added?

4. When the value of one unknown is found, why is it substituted for that unknown? Why by preference in one of the given equations?

5. In example ($B$) why are equations (1) and (2) multiplied respectively by 8 and 5? Unless one multiplies by the given coefficients, as 56 and 35, how shall one know by what numbers to multiply?

6. If one wished to obtain from equations (1) and (2) an equation without $y$, by what numbers should (1) and (2) be multiplied?

7. Why is equation (4) subtracted from equation (3)? If $y$ was to be removed, would (4) be subtracted from (3)?

8. Does the proof show that the same values of $x$ and $y$ will satisfy both equations?

9. What is the meaning of *simultaneous?* Of "simultaneous equations"?

10. If adding two equations causes two terms which contain one of the unknown quantities to become zero, what has been done in regard to this unknown?

11. What is the meaning of the word *eliminate?* What is its derivation?

12. If $5x - 3y = 11$ is added to $5x + 3z = 26$, will an unknown number disappear?

13. If $5x - 3y = 11$ is subtracted from $5x + 3z = 26$, can the value of either $y$ or $z$ be found from the new equation?

**14.** Can the value of two unknown numbers be found from one equation? Can the value of three unknown numbers be found from two equations?

**15.** If the coefficient of one term in an equation is multiplied, must all the terms in the member be multiplied by the same number? All in both members?

**16.** If the terms to be eliminated are $7x$ and $5x$, what must be done? If they are $6x$ and $3x$? If they are $28x$ and $63x$?

**17.** If the terms to be eliminated are $+5x$ and $-5x$, what must be done? If they are $+8y$ and $+8y$? If they are $-7x$ and $-7x$?

**18.** If the coefficients of the unknown to be eliminated are 96 and 60, what common factor could be rejected in finding multipliers? What factors prime to each other would be used if the coefficients were 85 and 68?

**19.** Given, $2x+y=11$ and $6x+3y=33$. What follows if an attempt is made to eliminate either unknown? Are these equations derived from independent conditions or is there but one equation in two forms?

**20.** In example ($A$) which equations express conditions of the problem? Which equation is derived from another? In example ($B$) which equations express conditions?

**21.** Which of the equations in the solutions of the two problems may be called independent? Dependent?

**22.** Given, $3x-2y=12$ and $3x+2y=7$. Is either of these equations derived from the other? Solve the equations. Now find by *trial* what positive whole numbers will satisfy each equation. How many values of $x$ and of $y$ in the two equations?

23. If positive integral values are required for $x$ and $y$, are the preceding equations consistent? If a person forms two inconsistent equations from a problem, where may the error be?

24. A man bought 3 live animals, $x$ cows and $y$ sheep. The number of cows less the number of sheep is 9. Are the equations satisfied by the results, $x=6$, $y=-3$? But the results are not reasonable. What may be done?

25. Write a rule for solving simple simultaneous equations having two unknown quantities.

## PROBLEMS

**51.** 1. If from seven times a father's age eleven times the age of his son is taken, the remainder is 154 years. How old is each if two times the son's age is equal to what the father's age was thirteen years ago?

2. Find the fraction which becomes equal to $\frac{3}{8}$ when 2 is added to the numerator, and equal to $\frac{1}{7}$ when 4 is added to the denominator.

3. Find the fraction that becomes equal to 2 when 15 is added to the numerator, and equal to $\frac{1}{2}$ when 3 is added to the denominator.

4. There is a number of two digits which is equal to seven times the sum of the digits. If from the number 18 is subtracted, the remainder will equal a number formed from the same digits in reverse order. Find the number.

5. If 9 is added to a number of two digits, the digits change places. The sum of the two numbers is 33. What are the digits? What are the numbers?

6. One-third of the sum of two numbers is 31, and three times their difference is one-third of 171. What are the numbers?

7. Two boys met on the highway. One had walked at the rate of 4 miles an hour, the other at the rate of $4\frac{1}{2}$ miles. They found that they started from towns 64 miles apart, and that if each had walked at the other's rate they would have accomplished one-half mile less. How many hours was each on the road?

8. How much apiece are oranges and apples when 15 oranges and 9 dozen apples cost \$1.92, and 3 dozen oranges and 50 apples at the same rate cost \$1.47?

9. Two girls have the same number of exercises. The first does twice as many as the second, and together they do 27. How many does each perform?

10. How many of each if a certain number of trout and perch equals 108, and one-half as many trout and 11 times as many perch make 75?

11. A storekeeper sells in one day 9 balls and 1 bat, and receives for them \$$2\frac{3}{4}$. The next day he receives \$2.25 for 3 balls and 3 bats. What is the price of each?

12. A girl has a fractional part of a dollar. If to the numerator of the fraction 3 is added, and from the denominator 3 is subtracted, the fraction will represent \$2. If 2 is added to both numerator and denominator, the fraction will represent 90 cents. How many cents has she?

13. If A can do a piece of work in 7 days, what part of it can he do in 1 day? What part can B do in 1 day if he can do the whole in 6 days? What part of the whole work can both do in 1 day? If they can do $\frac{13}{42}$ in one day, how long will it take them to do $\frac{42}{42}$? What does $\frac{42}{42}$, or 1, represent?

**14.** A can do a piece of work in $x$ days, B can do it in $y$ days. What part can they both do in one day? If they can together do it in 4 days, what part can they do in 1 day? The difference between the fractions representing what A and B can do in one day is $\frac{1}{12}$. How many days will each require if he does the work alone?

[In solving the equations of this problem is it better to eliminate without clearing the equations of fractions?]

**15.** A worked 24 days, B worked 12 days, and they finished a piece of work; but if A had worked 6 days and B 10 days, they could have done only one-half of it. In how many days could each have done it alone?

**16.** C and D can do a piece of work if C works 3 days and D works 2 days. If C works 4 days and D 1 day they will do $\frac{11}{12}$ of it. How many days does each require when working alone?

**17.** The playground contained 3500 square yards. It was enlarged by making it 5 yards longer and 3 yards broader. Its area was then 3975 square yards. The width of the old plus the length of the new playground is 125 yards. What were the original dimensions?

**18.** Find two consecutive even numbers the sum of whose reciprocals is $\frac{13}{84}$. The sum of 18 times one reciprocal plus 7 times the other equals 2.

**19.** Two men could do a piece of work in 3 days; but after they had worked together one day A left, and B finished the work in 4 days more. How many days does each require when working alone?

**20.** Two persons, C and D, could finish a work in $p$ days; they work together $q$ days, when C is called away, and D finishes it in $v$ days. In what time could each do it alone?

**21.** A boy bought 5 dozen oranges for $1.00. He sold them some at the rate of 2 for 5 cents and some at the rate of 3 for 10 cents. He gained 80 cents. How many of each kind did he sell?

**22.** There are ten consecutive odd numbers. The sum of the first and the last is 60; the sum of one-seventh of the first and one-third of the last is 16. What are the numbers? How many times the common difference in the second number? In the tenth number?

**23.** On her birthday a girl put some money into the savings bank. The next birthday she deposited another sum. The difference between seven times the first sum and two times the second sum was $27. When her money had gained one-fourth of itself she had $11.25. What were the sums deposited?

**24.** A brother and sister who had been to the "Zoo" found that the number of animals and birds which they had fed with peanuts was 77, and that $\frac{3}{11}$ of the number of animals plus $\frac{1}{8}$ of the number of birds was $\frac{1}{11}$ of the whole number. How many birds were there?

**25.** In the air, "Roll Jordan, Roll," the eighth notes are in number $\frac{1}{5}$ of the number of quarter notes. The number of eighth notes less the number of sixteenth notes equals the number of quarter notes less 2. There are two half notes, and the whole number of notes is 48. How many notes of each kind?

## 52. PROBLEMS LEADING TO THREE SIMULTANEOUS EQUATIONS

*Example A.* A girl who had three kinds of candy wished to give her sister 34 pieces. She tried three groups. In the first she put all of the first kind and one-half of the second and one-half of the third; in the second group she put all of the second kind and one-third of each of the other two; in the third group she put all of the third kind and one-fourth of each of the other two. How many pieces of candy had she?

Let $x$, $y$, and $z$ represent the number of pieces.

$$\begin{cases} x + \frac{y}{2} + \frac{z}{2} = 34 & (1) \\ \frac{x}{3} + y + \frac{z}{3} = 34 & (2) \\ \frac{x}{4} + \frac{y}{4} + z = 34 & (3) \end{cases}$$

$$\begin{aligned} 2x + y + z &= 68 && (4) \\ x + 3y + z &= 102 && (5) \\ x + y + 4z &= 136 && (6) \end{aligned}$$

$$\begin{aligned} 2x+6y+2z &= 204 && (7)\ \text{Eq. } (5)\times 2 \\ 2x+ y+ z &= 68 && (4) \\ \hline 5y+ z &= 136 && (8) \end{aligned}$$

$$\begin{aligned} x+3y+ z &= 102 && (5) \\ x+ y+4z &= 136 && (6) \\ \hline 2y-3z &= -34 && (9) \end{aligned}$$

$$\begin{aligned} 15y+3z &= 408 && (10)\ \text{Eq.}(8)\times 3 \\ 2y-3z &= -34 && (9) \\ \hline 17y &= 374 && (11) \\ y &= 22 && (12) \\ 44-3z &= -34 \\ -3z &= -78 \\ z &= 26 \\ x+11+13 &= 34 \\ x &= 10 \end{aligned}$$

Ver.: $10 + \frac{22}{2} + \frac{26}{2} = 34$

$\frac{10}{3} + 22 + \frac{26}{3} = 34$

$\frac{10}{4} + \frac{22}{4} + 26 = 34$

The girl had 58 pieces of candy.

In solving the equations of Example *A* two combinations are made for the purpose of eliminating $x$.

The terms of Eq. (5) are multiplied by 2, making Eq. (7).

From Eq. (7), Eq. (4) is subtracted, and there results Eq. (9) with two unknown quantities.

Eq. (6) is subtracted from Eq. (5) and Eq. (11) results.

The two new equations are combined and $z$ is eliminated.

The value of $y$ is substituted in Eq. (9) and the value of $z$ is found. The values of $y$ and $z$ are substituted in Eq. (1) and the value of $x$ is found.

*Example B.* The sum of the reciprocals of three numbers is $\frac{7}{10}$. If the second reciprocal is subtracted from the sum of the other two, the remainder is $\frac{3}{10}$. If the third reciprocal is subtracted from the sum of the first and second, the remainder is $\frac{11}{30}$. What are the numbers?

Let $x$, $y$, $z$ be the numbers.

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{10}$ (1)

$\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=\frac{3}{10}$ (2) Add Eqs. (1) and (2)

$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=\frac{11}{30}$ (3) Add Eqs. (2) and (3)

$\frac{2}{x}+\frac{2}{z}=1$ (4)

$\frac{2}{x}=\frac{2}{3}$ (5)

$x=3$ Subst. in (4)

$z=6$

Substitute values of $x$ and $z$ in (1)

$\frac{1}{3}+\frac{1}{y}+\frac{1}{6}=\frac{7}{10}$

$y=5$

The numbers are 3, 5, and 6.

In example *A* and example *B* the first three equations follow the order of the conditions of the problem, and the connection between the problem and its algebraic interpretation is apparent.

In the solution of problem *C*, page 106, the conditions are considered in reverse order. Is it apparent that equation (7) states the first condition of the problem, and that equation (4) states the second condition?

Rewrite this solution of *C*: state the conditions in their order; make (1), (2), (3) mean something more than order of writing; simplify the statements in order. Observe the equations that are statements, that are simplified statements, that are derived from statements or from statements simplified.

*Example C.* Find a number the sum of whose digits is 13. If 4 is subtracted from the number, and the remainder is divided by the sum of the two left-hand digits, the quotient is 20. If 297 is added to the number, the sum will be a number of the same digits reversed.

Let $x$, $y$, and $z$ represent the digits.

$100x + 10y + z =$ the number.

$100z + 10y + x =$ a number with the digits reversed.

$$100x + 10y + z + 297 = 100z + 10y + x \qquad (1)$$

$$99x - 99z = -297 \qquad (2)$$

$$x - z = -3 \qquad (3)$$

$$\frac{100x + 10y + z - 4}{x + y} = 20 \qquad (4)$$

$$100x + 10y + z - 4 = 20x + 20y \qquad (5)$$

$$\left.\begin{aligned} 80x - 10y + z &= 4 \qquad (6)\\ x + y + z &= 13 \qquad (7)\\ x - z &= -3 \qquad (3)\end{aligned}\right\}$$

$$79x - 11y = -9 \qquad (8)$$

$$2x + y = 10 \qquad (9)$$

$$x = 1$$

$$y = 8$$

$$z = 4 \qquad \text{The number is 184.}$$

## TEST Q

**1.** In general how many equations does a problem require for its solution?

**2.** How many combinations are made with three equations in order to eliminate one quantity? With two equations? With four equations? How many new equations result from combinations compared with the number of equations expressing the conditions of the problem?

3. Why must the same unknown be eliminated in each combination of the given equations?

4. Equations may be solved as many times as there are unknown quantities, by beginning first with one unknown and then with another. But there is generally a choice. See Example *A*, equations (4), (5), (6). Is there an advantage in beginning with any one? In eliminating $z$, which is a better combination, (1) and (3), (2) and (3); or, (1) and (2), (2) and (3)? Why? [Ex. *A*, page 104.]

5. In Example *C* is there a choice either in unknowns or in combinations?

### PROBLEMS

1. A number is expressed by three digits whose sum is 19. Reversing the order of the first two figures makes a number 180 less. Reversing the order of the last two figures makes a number greater by 18. What is the number?

2. A and B together can do a piece of work in $2\frac{2}{9}$ days. B and C can do it in $2\frac{2}{5}$ days, and A and C can do it in $2\frac{8}{11}$ days. How long will it take each alone?

3. If 9 bushels of oats, 3 of rye, and 2 of wheat are bought for \$7.25; 6 of oats, 5 of rye, and 3 of wheat, for \$8.75; and 3 of oats, 4 of rye, and 5 of wheat, for \$9, what is the price per bushel of each kind of grain?

### EXERCISES IN THE CONSTRUCTION OF PROBLEMS. III

**53.** To understand clearly any class of problems, one should endeavor to make the facts given seem *real*, and should consider the possibilities of the class.

(*A*) Suppose that a pond is a mile wide. How long did it take a boy to row across? What is his rate (miles an hour) in still water? How long did it take him to drift down the river from his boathouse to the bridge, the distance of a mile? What is the rate (miles an hour) of the river current? How long did it take the boy to row down to the bridge in this current? What distance could the two forces (the rower's and the stream's) accomplish in one hour? Going back, the boy's rate was the same as when he rowed on the pond; that is, he made the same effort in pulling at his oars, but the stream, at its rate, pushed in the opposite direction. What did the adverse forces accomplish in one hour?

Suppose that the boy rowed across the pond in fifteen minutes, then his rate is four miles an hour. If it is thought half an hour is a reasonable answer to the third question, then the rate of the current is two miles an hour. If both rower and current work in the same direction, they can take the boat six miles down stream in an hour; if rower and current work in opposite directions, the rower takes the boat up stream at the rate of two miles an hour.

There enter into a problem of this class, the distance; the time required to row one way, to row each way, or to row both ways; the rate of the current, understood to be uniform; the rate of the rower or of a crew, also supposed to be uniform.

### EXERCISES

Having the following numbers in mind, make as many problems as possible involving two simultaneous equations:

The rate of the crew, 9 miles an hour; the rate of the river current, 3 miles an hour; the distance in one direction, 12 miles; the time required to row down, 60 minutes; the time required to return, 120 minutes.

(*B*) Suppose that a hare had run across the field from the stone wall before the dog got to it, and that he followed until he caught her. The hare had made many short leaps; the dog had made longer leaps and fewer of them. When he caught her they had gone the same distance from the wall; he had measured the distance in his leaps, and she in hers.

If these leaps can be reduced to feet, there will be two expressions for the same distance, in the same denomination, available for an equation.

1. Suppose that the hare was 50 leaps ahead of the dog; that she took 4 leaps to the dog's 3; that 2 of his leaps were equivalent to 3 of hers (that is, one of his leaps was three-halves of one of hers).

Let $4x =$ the number of leaps the hare made when pursued.

$3x =$ the number of leaps the dog made.

$a =$ the number of feet in one of the hare's leaps.

$\frac{3a}{2} =$ the number of feet in one of the dog's leaps.

$50a =$ the number of feet the hare went before the dog started.

$50a + 4x \cdot a =$ the whole distance in feet.

$3x \cdot \frac{3a}{2} =$ the whole distance in feet.

$$\frac{9ax}{2} = 4ax + 50a.$$

$$9ax = 8ax + 100a.$$

$$x = 100.$$

$4x = 400$, the number of leaps the hare made while she was pursued.

$3x = 300$, the number of leaps the dog made in catching the hare.

2. A hare is ahead of a dog a distance equivalent to 90 of the dog's leaps. The hare makes 5 leaps to the dog's 4, but 3 of his leaps are equivalent to 4 of hers. How many leaps did each make before the hare was caught?

Let $4x =$ the number of leaps the dog makes.

$5x =$ the number of leaps the hare makes in the same time.

$a =$ the number of feet in one leap of the hare.

$\frac{4a}{3} =$ the number of feet in one leap of the dog.

$90 \times \frac{4a}{3} + 5ax =$ the whole distance in feet.

$4x \cdot \frac{4a}{3} =$ the whole distance in feet.

$$\frac{16ax}{3} = 5ax + 120a.$$

$$16ax = 15ax + 360a.$$

$$x = 360.$$

$4x = 1440$, the number of leaps the dog makes.

$5x = 1800$, the number of leaps the hare makes when she is pursued.

In the following condensed problems are: (1) the number of leaps the hare is ahead of the dog; (2) the ratio of leaps, the hare's number always the larger; (3) the equivalence of leaps, the dog's number the smaller; (4) the number of times one hare-leap in one leap of the dog.

| | | | | |
|---|---|---|---|---|
| 1. | Hare's, 50; | 5 : 3; | $1 \Leftrightarrow 2$; | 2. |
| 2. | 30; | 4 : 3; | $3 \Leftrightarrow 5$; | $\frac{5}{3}$. |
| 3. | 50; | 7 : 5; | $5 \Leftrightarrow 8$; | $\frac{8}{5}$. |
| 4. | 120; | 5 : 3; | $1 \Leftrightarrow 2$; | 2. |
| 5. | 60; | 6 : 5; | $7 \Leftrightarrow 9$; | $\frac{9}{7}$. |
| 6. | 60; | 9 : 6; | $3 \Leftrightarrow 7$; | $\frac{7}{3}$. |
| 7. | 63; | 4 : 3; | $5 \Leftrightarrow 9$; | $\frac{9}{5}$. |
| 8. | 60; | 3 : 2; | $3 \Leftrightarrow 7$; | $\frac{7}{3}$. |
| 9. | Dog's, 80; | 3 : 2; | $1 \Leftrightarrow 2$; | 2. |
| 10. | Dog's, 48; | 5 : 4; | $5 \Leftrightarrow 7$; | $\frac{7}{5}$. |

## EXERCISES

1. Write the equation directly from several of the skeleton problems.

2. What would be the result, the other conditions remaining, if the leaps of the hare and the dog were equal? If the number of leaps was the same?

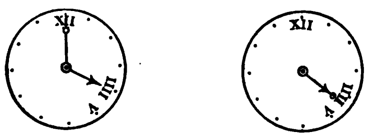
3. Write problems from skeletons 7, 8, 9, 10, omitting (4), the number of times one hare-leap in one leap of the dog.

4. Construct other problems having similar conditions. Consider the number of the strokes of two rowers in a given time, the number of equivalent strokes in a given distance, etc.

Construct any problem of pursuit.

(*C*) *Example* 1. At how many minutes after IIII will the hands of a clock be together?

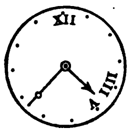
Let $M$ represent the minute hand, and $H$ the hour hand.

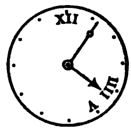


(1) How far apart in minute-spaces are $M$ and $H$ at IIII? (2) How much faster than $H$ does $M$ move? (3) If $x$ represents the number of minute-spaces $M$ passes from its position at XII until the hands are together, what will represent the number of minute-spaces $H$ passes over in the same time? (4) If this fraction is added to 20, what will the sum represent? (5) What has been made to represent the same distance? (6) Express the equality.

*Example* 2. At how many minutes after IIII o'clock will the hands of a clock be at right angles, or fifteen minute-spaces apart?

There will be two solutions: the hands will be fifteen minute-spaces apart before $M$ passes $H$, and again after it passes $H$.




Let $x$ equal the number of minute-spaces $M$ passes before the hands are at right angles, and let $\frac{x}{12}$ represent the number of minute-spaces $H$ passes in the same time. Then $M$ is $x$ minute-spaces from XII, and $H$ is $20 + \frac{x}{12}$ minute-spaces from XII. The difference between their positions is 15 minute-spaces; hence, $x = 20 + \frac{x}{12} - 15$.

When $M$ has passed $H$, it will be 15 minute-spaces farther from XII than $H$ will be; hence, $x = 20 + \frac{x}{12} + 15$.

(*a*) $x = 20 + \frac{x}{12} - 15$ (1)

$12x = 240 + x - 180$ (2)

$x = 5\frac{5}{11}$ min. after IIII.

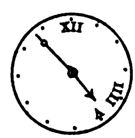
(*b*) $x = 20 + \frac{x}{12} + 15$ (1)

$12x = 240 + x + 180$ (2)

$x = 38\frac{2}{11}$ min. after IIII.

*Example* 3. At how many minutes after IIII will the hands of a clock be opposite each other?

Let $x$ equal the number of minute-spaces $M$ passes, and let $\frac{x}{12}$ equal the number of minute-spaces $H$ passes from the IIII o'clock position before the hands are opposite each other. $H$ is $20 + \frac{x}{12}$ minute-spaces from XII, $M$ is $x$ minute-spaces from XII, and there are 30 minute-spaces between them; hence,




$$x = 20 + \frac{x}{12} + 30.$$

From this equation $x = 54\frac{6}{11}$, the number of minutes after IIII.

Let $a$ represent the number of minutes between the hands at the hour named; then, a general formula for finding the number of minutes after the hour (1) when the hands will be together is $x = a + \frac{x}{12}$;

(2) when the hands will be at right angles, two cases, $x = a + \frac{x}{12} \pm 15$;
(3) when the hands will be opposite, one case, $x = a + \frac{x}{12} \pm 30$.

There are special instances to be considered:

1. At what time between XII and I are the hands of a clock together? $x = 0 + \frac{x}{12}$.

2. At what time between III and IIII are the hands of a clock at right angles? $x = 15 + \frac{x}{12} \pm 15$.

3. At what time between VI and VII are the hands of a clock opposite each to the other? $x = 30 + \frac{x}{12} \pm 30$.

(1) How is the result, $x = \frac{x}{12}$, to be interpreted?

(2) Should the hands move at the same rate in order to make the statements of problems 1 and 3, and one case in 2, possible?

(3) From $x = 30 + \frac{x}{12} + 30$, $x = 65\frac{5}{11}$. At what time, then, after VI are the hands opposite?

4. At what time between I and II, second case, are the hands of a clock at right angles?

(1) The hour-hand is how many minute-spaces from XII?

(2) How many additional minute-spaces must the minute-hand pass that there may be fifteen spaces between the hands? $x = 5 + \frac{x}{12} + 45$.

5. At what time between II and III are the hands of a clock at right angles, second case? Solve and explain,

$$x = 10 + \frac{x}{12} + 45.$$

6. At what time between IX and X are the hands of a clock at right angles? How many cases?

7. At what time between X and XI are the hands of a clock at right angles? Solve and explain,

$$x = 50 + \frac{x}{12} \pm 15.$$

8. At what time between XI and XII are the hands of a clock at right angles? $x = 55 + \frac{x}{12} \pm 15.$

9. At what time between XII and I are the hands of a clock at right angles? Explain, $x = 0 + \frac{x}{12} + 45.$

10. Write all the equations required in finding at what time after each hour the hands of a clock are opposite each other.

When should 30, the number of minute-spaces between the hands, be negative in these equations?

11. At what time between IX and X are the hands of a clock together?

12. At what time between VI and VII is the minute hand of a clock 5 minutes in advance of the hour-hand?

13. At what time between X and XI is the minute-hand of a clock 25 minutes in advance of the hour-hand?

14. At what time between V and a half hour after V are the hands of a clock at right angles to each other?

15. At what time between II and III is the hour-hand of a clock one minute in advance of the minute-hand?

# X. Problems Leading to Quadratic Equations

**54.** *Example A.* Find the length and the breadth of a garden whose area is 6480 square feet, and whose length is to its breadth as 5 : 4.

Let $5x$ = the number of yards in the length.

$4x$ = the number of yards in the breadth.

$$20x^2 = 6480 \qquad (1)$$

$$x^2 = 324 \qquad (2) \quad \text{Ax. R.}$$

$$x = \pm 18 \qquad (3)$$

$$\left.\begin{aligned} 5x &= 90 \\ 4x &= 72 \end{aligned}\right\} \qquad \begin{aligned} (4) \\ (5) \end{aligned}$$

(1) Is equation (2) a simple equation? Of what degree is it? Why is it called quadratic? May it be called pure quadratic?

(2) Why is not the sign $\pm$ given to $x$ as well as to 18? Write all the values of $+x$ and of $-x$ if $\pm x = \pm 18$. Now divide the two equations which give the values of $-x$ by $-1$. (Ax. D.) How many values of $x$ are there then? Is it necessary to give $x$ the $\pm$ sign?

(3) Why take $+18$ only as the value of $x$ in getting the dimensions of the garden?

*Example B.* Subtract 1 from a certain number; add 1 to the number; divide the difference by the sum, and the sum by the difference. The sum of the quotients is $\frac{5}{2}$. What is the number?

Let $x =$ the number.

$$\frac{x-1}{x+1}+\frac{x+1}{x-1}=\frac{5}{2} \qquad (1)$$

$$(x-1)^2+(x+1)^2=\frac{5(x^2-1)}{2} \qquad (2)$$

$$4(x^2+1)=5(x^2-1) \qquad (3)$$

$$4x^2+4=5x^2-5 \qquad (4)$$

$$x^2=9 \qquad (5) \quad \text{Ax. R.}$$

$$x=\pm 3 \qquad (6)$$

Verification:

$$\frac{+3-1}{+3+1}+\frac{+3+1}{+3-1}=\frac{5}{2} \qquad \frac{-3-1}{-3+1}+\frac{-3+1}{-3-1}=\frac{5}{2}$$

$$\frac{2}{4}+\frac{4}{2}=\frac{5}{2} \qquad \frac{-4}{-2}+\frac{-2}{-4}=\frac{5}{2}$$

$$\frac{1}{2}+2=\frac{5}{2} \qquad 2+\frac{1}{2}=\frac{5}{2}$$

Both $+3$ and $-3$ satisfy the equation. Does each number satisfy the conditions of the problem?

## AFFECTED QUADRATIC EQUATIONS

**55.** *Example C.* A man bought some cows for \$144. Twelve more cows for the same money would have cost \$1 less apiece. How many cows did he buy?

Let $x =$ the number bought.

$\frac{144}{x} =$ the price paid for one cow.

$\frac{144}{x+12} =$ the price paid if there were 12 more.

$$\frac{144}{x}-\frac{144}{x+12}=1 \qquad (1)$$

$$144x+1728-144x=x^2+12x \qquad (2)$$

$$x^2+12x=1728 \qquad (3)$$

The conditions of this problem produce an affected quadratic equation, an equation which contains the second and the first powers of the unknown. That $x$ may stand alone, equal to some number which is its value in the equation, a number which satisfies the conditions of the problem, the square root must be taken. (Ax. R.) But the first member is not a square. It can be made a square. (Ax. A.)

The following examples call attention to what fact in the structure of a trinomial square?

1. $x^2 + 2yx + y^2 = x^2 + 2y \cdot x + \left(\frac{2y}{2}\right)^2.$

2. $x^2 + 4x + 4 = x^2 + 4 \cdot x + (\frac{4}{2})^2.$

3. $x^2 + \frac{2}{3}x + \frac{1}{9} = x^2 + \frac{2}{3} \cdot x + (\frac{1}{2} \times \frac{2}{3})^2.$

4. $x^2 + \frac{5}{8}x + \frac{25}{256} = x^2 + \frac{5}{8} \cdot x + (\frac{1}{2} \times \frac{5}{8})^2.$

## EXERCISE 46

Add a term to each of the following numbers, and make of each a trinomial square, expressing the third term as in the above examples:

1. $x^2 + \frac{3}{2}x$
2. $x^2 - 7x$
3. $x^2 - \frac{1}{5}x$
4. $x^2 - \frac{7}{6}x$
5. $x^2 + \frac{5}{2}x$
6. $x^2 - \frac{25x}{6}$
7. $x^2 - \frac{30x}{9}$
8. $x^2 + \frac{39x}{10}$
9. $x^2 - 2x$
10. $x^2 - \frac{11x}{5}$
11. $x^2 - 100x$
12. $x^2 - 68x$
13. $x^2 + 2500x$
14. $x^2 + \frac{7x}{12}$
15. $x^2 - \frac{23x}{6}$
16. $x^2 - ax$
17. $x^2 + b^2x$
18. $x^2 - 2cx$
19. $x^2 - \frac{4d}{3}x$
20. $x^2 - (a+b)x$
21. $x^2 + (p-q)x$

## EXERCISE 47

Complete the square and preserve the equation:

*Example.* $x^2 - \frac{4x}{3} = \frac{55}{3}$

$$x^2 - \frac{4x}{3} + \left(\frac{2}{3}\right)^2 = \frac{55}{3} + \left(\frac{2}{3}\right)^2$$

1. $x^2 + \frac{x}{2} = \frac{15}{2}$
2. $x^2 + \frac{3x}{5} = \frac{2}{5}$
3. $x^2 + \frac{2x}{3} = 40$
4. $x^2 - \frac{4x}{3} = \frac{4}{3}$
5. $x^2 - \frac{x}{6} = \frac{1}{3}$
6. $x^2 - \frac{11x}{12} = -\frac{1}{6}$
7. $x^2 - \frac{2x}{15} = \frac{1}{15}$
8. $x^2 - \frac{44x}{3} = -\frac{121}{3}$
9. $x^2 - \frac{25x}{6} = -\frac{7}{2}$
10. $x^2 + \frac{17x}{4} = \frac{15}{4}$

## EXERCISE 48

Complete the square, preserve the equality, and simplify:

*Example.* $x^2 - \frac{17x}{3} = 2$

$$x^2 - \frac{17x}{3} + \left(\frac{17}{6}\right)^2 = 2 + \left(\frac{17}{6}\right)^2$$

$$x^2 - \frac{17x}{3} + \left(\frac{17}{6}\right)^2 = \frac{72}{36} + \frac{289}{36}, \text{ or } \frac{361}{36}$$

1. $x^2 + 2x = 8$
2. $x^2 - 6x = 7$
3. $x^2 - 4x = 12$
4. $x^2 + 4x = 5$
5. $x^2 + 5x = 14$
6. $x^2 + \frac{14x}{5} = 11$
7. $x^2 + \frac{22x}{3} = 2\frac{5}{9}$
8. $x^2 + \frac{x}{8} = \frac{30}{8}$
9. $x^2 - \frac{29x}{12} = -\frac{7}{6}$
10. $x^2 + \frac{x}{20} = \frac{3}{5}$

## EXERCISE 49

Take the square root of both members and find the values of $x$.

*Example.*

$$x^2+\frac{3x}{2}+\left(\frac{3}{4}\right)^2=\frac{289}{16}$$

$$x+\frac{3}{4}=\pm\frac{17}{4}$$

$$\left.\begin{aligned} x&=+\frac{17}{4}-\frac{3}{4},\text{ or }\frac{7}{2}\\ x&=-\frac{17}{4}-\frac{3}{4},\text{ or }-5\end{aligned}\right\}$$

1. $x^2-4x+(2)^2=9$
2. $x^2-10x+(5)^2=1$
3. $x^2-\frac{4x}{3}+\left(\frac{2}{3}\right)^2=\frac{169}{9}$
4. $x^2-6x+(3)^2=-4$
5. $x^2-2x+1=16$
6. $x^2+19x+\left(\frac{19}{2}\right)^2=\frac{1521}{4}$
7. $x^2-4x+(2)^2=4$
8. $x^2-\frac{35x}{3}+\left(\frac{35}{6}\right)^2=\frac{841}{36}$
9. $x^2+\frac{10x}{3}+\left(\frac{5}{3}\right)^2=\frac{121}{9}$
10. $x^2+\frac{11x}{5}+\left(\frac{11}{10}\right)^2=\frac{361}{100}$
11. $x^2-7x+\left(\frac{7}{2}\right)^2=\frac{81}{4}$
12. $x^2+14x+(7)^2=81$
13. $x^2-\frac{23x}{6}+\left(\frac{23}{12}\right)^2=\frac{1369}{144}$
14. $x^2-25x+\left(\frac{25}{2}\right)^2=\frac{7225}{4}$

## EXERCISE 50

Solve the given equations:

1. $x^2-x=72$
2. $x^2-9x=220$
3. $x^2-\frac{2x}{3}=32$
4. $x^2+\frac{19x}{5}=\frac{4}{5}$
5. $x^2-\frac{8x}{5}=\frac{21}{5}$

6. $x^2+\frac{4x}{3}=5$

7. $x^2+\frac{7x}{2}=\frac{15}{2}$

8. $x^2-\frac{x}{8}=\frac{7}{8}$

9. $x^2-11x=0$

10. $x^2-3x=-2$

11. $x^2-2x=8$

12. $x^2-5x=-6$

13. $x^2-x=6$

14. $x^2-4x=5$

15. $x^2+5ax=-6a^2$

16. $x^2+px=6p^2$

17. $x^2-2ax=b^2-a^2$

**56.** *Example A.* Ten times a number minus the square of the number equals 21. Find the number.

Let $x=$ the number.

$$10x-x^2=21 \quad (1) \qquad \text{or} \qquad +x^2-10x=-21 \quad (2)$$

The square root of $-x^2$ cannot be taken; hence the members of the equation must be divided by $-1$. (Ax. D)

*Example B.* Three times the square of a number plus seven times the number equals 110. Find the number.

Let $x=$ the number.

$$3x^2+7x=110 \quad (1) \qquad x^2+\frac{7x}{3}=\frac{110}{3} \quad (2)$$

The method of solving an affected quadratic equation taught in the preceding exercises requires that $x^2$ shall be positive, and its coefficient 1. Hence the members of equation (1) are divided by 3.

### PROBLEMS LEADING TO QUADRATIC EQUATIONS

1. A certain number of barrels of apples cost \$48. If there had been 4 barrels less for the money, each barrel would have cost \$1 more. How many barrels were there?

2. Two boys can do a piece of work in one-fifth of an hour. The older boy requires ten minutes less time than the younger requires to do the work alone. How many minutes does the younger need?

3. How many U.S. senators were there in 1796? If the number of senators less 7 is multiplied by 1 plus one-fourth of the number, the product is 225.

4. The square of the number of men from Pennsylvania who signed the Constitution of the United States equals three times the number plus 40. How many men were there?

5. Five times the square of the number of Virginia men who signed the Declaration of Independence plus three-sevenths of the number equals 248. How many Virginia men?

6. The square of the number of thousands of Indians in the United States plus twice the number of thousands equals 63,000. How many thousands are there?

7. Twenty times the number of letters in the title of a newspaper minus the square of the number equals 96. How many letters in the name?

8. The United States bought land of France for a certain number of millions of dollars. One hundred times the number minus four times the square of the number is 600. How many millions were paid?

9. If forty-seven sixths of the sum on the face of a piece of continental currency is added to the square of that sum, the amount will be $\$1\frac{1}{3}$. What is the face of the note?

10. What was the number of stars on the United States flag July 4, 1836? Four times the square of the number minus twenty times the number is 2000.

11. A pony travelled 30 miles at his ordinary rate; but if he had gone one-seventh of a mile more per hour he would have gained 10 minutes in the journey. His rate?

12. Find two numbers whose sum is 20, and whose product is $-525$.

13. There is a number whose digits, beginning with units, are consecutive. The product of the units' and tens' figures, plus the product of the tens' and hundreds' figures, equals the sum of the figures plus 20. What is the number?

14. The denominator of a given fraction is greater than the numerator by 2. If 5 is added to each term, the second fraction will be greater than the first by one-fifth. What is the given fraction?

15. A man increased his rate of walking by 3 furlongs; again he increased his first rate by 6 furlongs. He found that by the third rate he could walk 18 miles in eight-fifteenths of an hour less than by the second rate. How many furlongs an hour was his first rate?

16. The square of one-sixth of U. S. Grant's salary in 1861 taken from twenty times the salary leaves $2000. How many dollars did he receive a year?

17. How many years after 1800 was Boston made a city? Three times the number of years added to the square of the number equals 550.

18. How many colleges were there in Massachusetts when twenty times the number less the square of the number was 84?

19. What was the weight of "the largest piece of gold ever found in California," if the square of one-fifteenth of its weight less three-fifteenths of its weight was 130 pounds?

(Let $15x =$ the number of pounds the gold weighed.)

**57.** (1) Given, $(x-2)(x+5)=0$

The product being zero, one of the factors is zero or both factors are zero.

If $x-2=0,\ x=2.$

If $x+5=0,\ x=-5.$

(2) Given,

$$x^2-8x=-15 \qquad (1)$$

$$x^2-8x+15=0 \qquad (2) \quad \text{Ax. A}$$

$$(x-3)(x-5)=0 \qquad (3)$$

Factor the first member of (2). Suppose each factor in turn to be equal to 0.

$$x-3=0,\ x=3.$$

$$x-5=0,\ x=5.$$

Substitute 3 for $x$ in (1)

$$9-24=-15$$

Substitute 5 for $x$ in (1)

$$25-40=-15$$

Hence the results satisfy the equation.

**EXERCISE 51**

Solve by factoring:

1. $x^2+5x=24$
2. $x^2-5x=14$
3. $x^2-6x=27$
4. $x^2+7x=-10$
5. $-x^2+9x=+20$
6. $-x^2+3x=+2$
7. $x^2-4x=5$
8. $-(x^2-7x+12)=0$
9. $x^2-8x=-15$
10. $x^2-10x=11$
11. $2x^2-6x-20=0$
12. $34-16x^2=-2$
13. $2x^2-28x+90=0$
14. $5x^2+55x+120=0$
15. $(3x+2)(5x-3)=0$
16. $(x^2-9)(4x^2-16)=0$

# XI. Miscellaneous

## LITERAL, FRACTIONAL, AND NEGATIVE INDICES

**58.** (1) $$a^2 \times a^3 = a^{2+3}$$

$$a^m \times a^n = a^{m+n}$$

That is, $a$ to $m$ factors times $a$ to $n$ factors equals $a$ to $m + n$ factors.

(2) $$a^5 \div a^7 = a^{5-7} \text{ or } a^{-2}$$

$$a^{-5} \div a^{-2} = a^{-5-(-2)} \text{ or } a^{-3}$$

$$a^m \div a^{-2m} = a^{m-(-2m)} \text{ or } a^{3m}$$

(3) $$a^m \times a^{-m} = a^{m+(-m)} \text{ or } a^0$$

$$a^m \div a^m = a^{m-m} \text{ or } a^0$$

(4) $$a^m \times a^0 = a^{m+0} \text{ or } a^m$$

$$a^m \times 1 = a^m \qquad \therefore\ a^0 = 1$$

$$a^2 \div a^2 = a^0$$

$$a^2 \div a^2 = 1 \qquad \therefore\ a^0 = 1$$

(5) If $a^1$ is the product of two equal factors, the exponent 1 is the sum of two equal exponents, and $a^1 = a^{\frac{1}{2}+\frac{1}{2}}$.

Also, $a^1 = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$ and $a^1 = a^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}$.

Then $a^{\frac{1}{2}}$ is the square root of $a$; $a^{\frac{1}{3}}$ is the cube root of $a$; $a^{\frac{1}{4}}$ is the fourth root of $a$.

$$a^{\frac{5}{2}} \times a^{\frac{5}{2}} = a^{\frac{10}{2}} \text{ or } a^5$$

$a^{\frac{5}{2}}$ is one of two equal factors that produce $a^5$; hence, $a^{\frac{5}{2}}$ is the square root of $a^5$.

$$a^{\frac{1}{2}} = \sqrt{a};\ a^{\frac{1}{3}} = \sqrt[3]{a};\ a^{\frac{5}{2}} = \sqrt{a^5}$$

The numerator of the fractional exponent is the exponent of power, and the denominator is the index of root.

(6)
$$a^{\frac{1}{3}} \times {}^{\frac{1}{2}} = a^{\frac{1}{3}+\frac{1}{2}} \text{ or } a^{\frac{5}{6}}$$
$$a^{\frac{1}{4}} \times a^{-\frac{1}{2}} = a^{\frac{1}{4}+(-\frac{1}{2})} \text{ or } a^{-\frac{1}{4}}$$
$$a^{\frac{1}{2}} \times a^{-\frac{1}{2}} = a^{\frac{1}{2}+(-\frac{1}{2})} \text{ or } a^0$$
$$a^{\frac{2}{5}} \div a^{-\frac{2}{5}} = a^{\frac{2}{5}-(-\frac{2}{5})} \text{ or } a^{\frac{4}{5}}$$
$$a^{-\frac{1}{2}} \div a^{-\frac{1}{4}} = a^{-\frac{1}{2}-(-\frac{1}{4})} \text{ or } a^{-\frac{1}{4}}$$
$$a^{-\frac{3}{5}} \div a^{-\frac{3}{5}} = a^{-\frac{3}{5}-(-\frac{3}{5})}, \text{ or } a^0, \text{ or } 1$$

(7)
$$\frac{a^0}{b^3} = \frac{1}{b^3}. \quad a^0 b^3 = b^3. \quad a^0 + b^3 = 1 + b^3$$

$a^0$ as a factor may be discarded; $a^0$ as a term must be written, or 1 must be substituted for it.

(8)
$$\frac{1}{a^{-2}} \times \frac{a^2}{a^2} = \frac{a^2}{a^0} \text{ or } a^2$$
$$\frac{1}{a^2} \times \frac{a^{-2}}{a^{-2}} = \frac{a^{-2}}{a^0} \text{ or } a^{-2}$$

That is, the reciprocal of a quantity may be written in the form of a whole number if the sign of the exponent of the quantity as a whole is changed, or if the signs of the exponents of its factors are changed.

The change depends upon the law that both numerator and denominator of a fraction may be multiplied by the same number without loss or increase of value.

Ex. ($a$). $\frac{1}{a^2b^3} \times \frac{a^{-2}b^{-3}}{a^{-2}b^{-3}} = \frac{a^{-2}b^{-3}}{a^0b^0}$ or $a^{-2}b^{-3}$

Ex. ($b$). $\frac{1}{a^2+b^3} \times \frac{(a^2+b^3)^{-1}}{(a^2+b^3)^{-1}} = \frac{(a^2+b^3)^{-1}}{(a^2+b^3)^0}$ or $(a^2+b^3)^{-1}$

Ex. ($c$). $\frac{1}{(a+b)^2(a-b)} \times \frac{(a+b)^{-2}(a-b)^{-1}}{(a+b)^{-2}(a-b)^{-1}} = ?$

(9) Any fraction may be written in the form of a whole number.

$$\frac{5\,a^2}{3\,d^3m^2} \times \frac{3^{-1}d^{-3}m^{-2}}{3^{-1}d^{-3}m^{-2}} = \frac{5 \cdot 3^{-1}a^2d^{-3}m^{-2}}{3^0d^0m^0} \text{ or } 5 \cdot 3^{-1}a^2d^{-3}m^{-2}$$

(10) Any fraction may be freed from negative exponents.

$$\frac{a^{-4}}{b^{-5}} \times \frac{a^4b^5}{a^4b^5} = \frac{a^0b^5}{a^4b^0} \text{ or } \frac{b^5}{a^4}$$

This process is not inversion. It does not alter the value of the fraction.

$$\frac{a^{-2}+b^{-3}}{c^{-4}+d^{-5}} \times \frac{a^2b^3c^4d^5}{a^2b^3c^4d^5} = \frac{b^3c^4d^5 + a^2c^4d^5}{a^2b^3d^5 + a^2b^3c^4}$$

(11) $(x^3)^m = x^{3+3+3 \cdots \text{ to } m \text{ terms}}$ or $x^{3m}$

$(3\,a^2)^2 = 3^2\,a^4$ or $9\,a^4$

$(3\,a^2)^m = 3^m\,a^{2m}$

$(-x)^n = -x^n$ if $n$ is an odd number

$(-x)^n = +x^n$ if $n$ is an even number

$(5\,a^m)^{-\frac{1}{n}} = 5^{-\frac{1}{n}}a^{-\frac{m}{n}}$

Only odd powers of negative monomials have the minus sign.

(12) $(a^2)^3 = a^{2+2+2}$ or $+a^6$. $(a^3)^2 = a^{3+3}$ or $+a^6$.

$\sqrt[3]{a^6}$ or $\sqrt[3]{a^{2+2+2}} = +a^2$. $\sqrt[3]{-a^6} = -a^2$.

$\sqrt{a^6} = \pm a^3$. $\sqrt{-a^6}$. The root is imaginary.

$\sqrt[3]{x^{3m}} = +x^m$. $\sqrt[m]{x^{3m}} = +x^3$ if $m$ is odd.

$\sqrt[3]{-x^{3m}} = -x^m$. $\sqrt[m]{-x^{3m}} = -x^3$ if $m$ is odd.

If the index is an odd number, the root has the same sign as the quantity. If the index is an even number, and the quantity is positive, the root has the sign $\pm$; if the quantity is negative, the root is imaginary. No negative number is the product of an even number of negative factors.

## 59. HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

*Example.* Find the H. C. F. of the entire numbers,

$$4x^5 + 14x^4 + 20x^3 + 70x^2$$

and
$$8x^7 + 28x^6 - 8x^5 - 12x^4 + 56x^3$$

or of
$$2x^2(2x^3 + 7x^2 + 10x + 35)$$

and
$$4x^3(2x^4 + 7x^3 - 2x^2 - 3x + 14)$$

$$\begin{array}{r|l} 2x^3 + 7x^2 + 10x + 35 \,) & 2x^4 + 7x^3 - 2x^2 - 3x + 14 \,(x \\ & \underline{2x^4 + 7x^3 + 10x^2 + 35x} \\ & -2\,)\,\underline{-12x^2 - 38x + 14} \\ & 6x^2 + 19x - 7 \end{array}$$

$$
\begin{array}{r|l}
6x^2+19x-7 & 2x^3+7x^2+10x+35\ (x+1 \\
 & \underline{3} \\
 & 6x^3+21x^2+30x+105 \\
 & \underline{6x^3+19x^2-7x} \\
 & \qquad 2x^2+37x+105 \\
 & \qquad \underline{3} \\
 & \qquad 6x^2+111x+315 \\
 & \qquad \underline{6x^2+19x-7} \\
 & \qquad\quad 46\ )\ 92x+322 \\
 & \qquad\qquad\quad 2x+7
\end{array}
$$

$$
\begin{array}{r|l}
2x+7 & 6x^2+19x-7\ (3x-1 \\
 & \underline{6x^2+21x} \\
 & \quad -2x-7 \\
 & \quad \underline{-2x-7} \\
 & \qquad 0
\end{array}
$$

H. C. F. of $2x^2$ and $4x^3$ is $2x^2$. H. C. F. of given numbers, $2x^2(2x+7)$.

In finding the H. C. F. of two algebraic expressions:

(1) The given numbers are arranged according to the powers of $x$, beginning with the highest power.

(2) The highest monomial factor is removed from each of the given numbers.

(3) Any dividend may be multiplied by a number which is not a factor of the corresponding divisor.

(4) Any remainder may be divided by a number which is not a factor of the preceding divisor.

(5) The highest common factor of the monomial factors removed from the given numbers, times the last divisor, is the H. C. F. of the given numbers.

### EXERCISE 52

Find the indicated products; then find the H. C. F. of the products:

1. $(2x-5)(3x+1) \quad = ?$
   $(2x-5)(9x^2-3x-1) = ?$
2. $a(2a+b)(a-2b) = ?$
   $a(2a+b)(a+3b) = ?$
3. $(x-2)(2x+x+4) \quad = ?$
   $c(x-2)(3x^2-x+2) = ?$
4. $m(2m^2-mn+n^2)(3m-2n) = ?$
   $n(2m^2-mn+n^2)(m+3n) \quad = ?$
5. $2a(x^2-2x+4)(x+6) \quad = ?$
   $4a^2(x^2-2x+4)(x^2+x-2) \quad = ?$

### EXERCISE IN CONSTRUCTION

Construct exercises in H. C. F., taking exercises 1–5 as guides. The factors should have the same letter of arrangement in the same order of its powers.

### EXERCISE 53

Find the H. C. F. of the following numbers:

1. $b^3+b^2-2,\ b^3+2b^2-3$
2. $2n^3-5n^2-5n+8,\ n^2-4n+3$
3. $2b^2+5b-3,\ 2b^3+b^2-5b+2$
4. $m^3+4m^2-4m-16,\ 2m^2-5m+2$
5. $3a^2b^4+3a^2b^3-6a^2b,\ 6ab^5+12ab^4-18ab^2$
6. $3n^3-7n^2+4n-4,\ 2n^3-3n^2+2n-8$

**60.** If one of two numbers is divided by their H. C. F., and the quotient is multiplied by the other of the two given numbers, the product will be their Lowest Common Multiple.

Exercises 1–5, § 59, furnish suggestions for the construction of exercises in L. C. M.

## 61. SQUARE ROOT

$$(x+y)^2 = x^2 + 2x \cdot y + y \cdot y, \quad \text{or} \quad x^2 + (2x+y)y$$

$$(x+4)^2 = x^2 + 2x \cdot 4 + 4 \cdot 4, \quad \text{or} \quad x^2 + (2x+4)4$$

The two forms of the trinomial square suggest the method of finding the square root of a polynomial.

*Ex. A.*

| | | | |
|---|---|---|---|
| | | $x^2 + 8x + 16 \mid x + 4$ | Root |
| | | $x^2$ | |
| Trial divisor, | $2x$ | $+8x + 16$ | $8x \div 2x = 4$ |
| Complete divisor, | $2x + 4$ | $+8x + 16$ | $(2x+4)4$ |
| | | $0$ | |

*Ex. B.*

| | | | |
|---|---|---|---|
| | | $n^2 - 16n + 64 \mid n - 8$ | Root |
| | | $n^2$ | |
| Trial divisor, | $2n$ | $-16n + 64$ | $-16n \div 2n = -8$ |
| Complete divisor, | $2n - 8$ | $-16n + 64$ | $(2n-8) \times -8$ |
| | | $0$ | |

*Ex. C.*

| | | |
|---|---|---|
| | | $16x^4 - 24x^3b + 25x^2b^2 - 12xb^3 + 4b^4 \mid 4x^2 - 3xb + 2b^2$ |
| | | $16x^4$ |
| Trial div., | $8x^2$ | $-24x^3b + 25x^2b^2$ |
| Comp. div., | $8x^2 - 3xb$ | $-24x^3b + 9x^2b^2$ |
| Trial div., | $8x^2 - 6xb$ | $+16x^2b^2 - 12xb^3 + 4b^4$ |
| Comp. div., | $8x^2 - 6xb + 2b^2$ | $+16x^2b^2 - 12xb^3 + 4b^4$ |
| | | $0$ |

**EXERCISE 54**

**62.** Simplify the complex and continued fractions:

Ex. (*a*). $\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x}}}$.

$$x+\frac{1}{x}=\frac{x^2+1}{x};\quad \frac{1}{\dfrac{x^2+1}{x}}=\frac{x}{x^2+1};\quad x+\frac{x}{x^2+1}=\frac{x^3+2x}{x^2+1};$$

$$\frac{1}{\dfrac{x^3+2x}{x^2+1}}=\frac{x^2+1}{x^3+2x}$$

Ex. (*b*). $\dfrac{a}{a-\dfrac{a+2}{a+2-\dfrac{a+1}{a}}}$.

$$a+2-\frac{a+1}{a}=\frac{a^2+a-1}{a}$$

$$\frac{a+2}{\dfrac{a^2+a-1}{a}}=\frac{a^2+2a}{a^2+a-1}$$

$$a-\frac{a^2+2a}{a^2+a-1}=\frac{a^3-3a}{a^2+a-1}$$

$$\frac{a}{\dfrac{a(a^2-3)}{a^2+a-1}}=\frac{a(a^2+a-1)}{a(a^2-3)},\text{ or }\frac{a^2+a-1}{a^2-3}$$

1. $1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{a}}}$

2. $\cfrac{1}{b+\cfrac{1}{1+\cfrac{b+1}{3-b}}}$

3. $1+\cfrac{a}{1+\cfrac{2a^3}{1-a^2}}$

4. $\cfrac{1}{a-\cfrac{a^2-1}{a+\cfrac{1}{a-1}}}$

5. $\cfrac{b}{c+\cfrac{a}{x+\cfrac{d}{n}}}$

6. $\cfrac{1}{1-\cfrac{1+n}{n-\cfrac{1}{n}}}$

7. $\cfrac{a-3}{a-3-\cfrac{a}{a-\cfrac{a-1}{a-2}}}$

8. $a+b-\cfrac{1}{a-b+\cfrac{ab}{a-b}}$

9. $\cfrac{n}{1-\cfrac{n}{n+1+\cfrac{n}{n^2-n+1}}}$

10. $\dfrac{\dfrac{b-1}{b}+\dfrac{c-1}{c}+\dfrac{d-1}{d}}{\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}}$

11. $\dfrac{\left(\dfrac{1}{x}+\dfrac{1}{y+z}\right)\left(\dfrac{y^2+z^2-x^2}{2yz}+1\right)}{\dfrac{1}{x}-\dfrac{1}{y+z}}$

12. $\dfrac{\left(\dfrac{1}{x^2}+\dfrac{1}{a^2}\right)\div\left(\dfrac{1}{x^2}-\dfrac{1}{a^2}\right)}{\left(\dfrac{1}{x}+\dfrac{1}{a}\right)\div\left(\dfrac{1}{x}-\dfrac{1}{a}\right)}$

# XII. Progressions

**63.** A series is a succession of numbers formed according to a law; the numbers are called the terms of the series; and if they increase or decrease by a common difference, they are in **arithmetical progression.**

The series 3, 5, 7, 9, 11, 13, 15 is a progression from 3 by the continued addition of 2. It might be written $3, 3+2, 3+2+2, 3+2+2+2, \cdots$, or in the form $3, 3+(2), 3+2(2), 3+3(2), 3+4(2), 3+5(2), 3+6(2)$.

Let $a$ represent 3, $d$ represent 2, $l$ represent 15. Then the series could be written $a, a+d, a+2d, a+3d, a+4d, a+5d, a+6d$.

### TEST R

1. From which term is the common difference absent?

2. How many times the c. d. in the 4th term?

3. In any term the coefficient of $d$ is how many units less than the number of the term?

4. Substitute the symbols in $3+6(2)$. Is there any letter for 6? The coefficient 6 is how many less than the number of the term? What represents the number of the term? What then will represent 6? What will represent the coefficient of $d$ in the last term of any series?

5. Take the series $15, 15+(-2), 15+2(-2), 15+3(-2), 15+4(-2), 15+5(-2), 15+6(-2)$. In this series what is $a$? $d$? $l$? $n$?

6. If $(n-1)$ is the coefficient of $d$ in the last term, what, by the same method of representation, is the coefficient of $d$ in next to the last term? In the second term? In the first term? What is its value in the first term?

7. If $l$ may always be represented by $a+(n-1)d$, what is the last term in a series whose $a=2$, $d=3$, and $n=6$.

8. What is the formula for the 5th term in any A. P.?

9. What is the 9th term of the given series beginning 3, 5, $\cdots$? What is the $n$th term in any series?

10. If two successive terms in an A. P. are 25 and 29, what is the common difference? If the successive terms are 2 and $-4$? 0 and $-a$? .3 and .5? $\frac{1}{2}$ and $\frac{3}{4}$?

11. Given $a$, $a+d$, $a+2d$, $a+3d$, $\cdots$. Add the first and third terms and divide their sum by 2. What term is the quotient like? Add the 2d and 4th terms and divide by 2. If half the sum of two successive odd terms is taken, what is produced? Half the sum of two successive even terms?

12. Find the mean or middle term, if the 1st and 3d terms are 15 and 7; 2 and $-10$; 7 and $-1$; $a-d$ and $a+3d$; $2a-b$ and $6a-5b$.

13. If the first term is 2 and the last is 23, what part of the last term is made up by additions of common difference? If there are 8 terms in all, why divide this part of 23 by 7 in order to find what the common difference is? What does $n-1$ stand for, if the last term is

$$2+\overline{3+3+3+3+3+3+3}?$$

14. Solve the equation $a+(n-1)d=l$, and find $a$; find $n$; find $d$.

**64.** If the number of terms is unknown, the representation must be incomplete; as,

$$a, \ a+d, \ a+2d, \ \cdots l$$

Reversing the series it may be written,

$$l, \ l-d, \ l-2d, \ \cdots a$$

The sum of the terms $= a + (a+d) + (a+2d) + \cdots + l$

The sum of the terms $= l + (l-d) + (l-2d) + \cdots + a$

Adding these two sums,

$$2S = (a+l) + (a+l) + \cdots \text{ to } n \text{ terms}$$

or, $$2S = n(a+l); \ S = \frac{n}{2}(a+l)$$

Take $$S = 3+5+7+9+11$$

and $$S = 11+9+7+5+3$$

$$2S = 14+14+14+14+14; \ S = \tfrac{5}{2}(14)$$

By solving the equation, $S = \frac{n}{2}(a+l)$, $a$, $l$, and $n$ may be found, each in terms of the other three letters. Also, $a+(n-1)d$ may be substituted for $l$ in the formula for $S$.

### EXERCISE 55

1. $a+2d=30$ and $a+5d=48$. Eliminate $a$ and find $d$. Eliminate $d$ and find $a$. Write the series if $a+5d$ is the $n$th term.

2. The 3d term of a series is 25; the 6th term is 40. What is the series?

3. The first term is 6; the common difference is 2; the number of terms is 8. What is the last term?

4. The last term is 22; the common difference is 2; the number of terms is 9. What is the first term?

5. The number of terms is 7; the first term is 5; the last term is 23. What is the common difference?

6. The common difference is 7; the last term is 36; the first term is 1. What is the number of terms?

7. What is the sum of the series when $a = 5$, $n = 19$, and $l = 77$?

8. What is the last term when $a = 12$, $n = 7$, and $S = 147$?

9. What is the first term when $l = 28$, $n = 8$, $S = 140$?

10. What is $S$ when $a = 3$, $d = 2$, $n = 8$?

11. What is $S$ when $a = 16$, $d = -3$, $n = 6$?

**65.** A series of numbers is in **geometric progression** if the quotients are the same when each term, beginning with the second, is divided by its predecessor.

The series 3, 6, 12, 24, 48 is a progression from 3 produced by the continued multiplying by 2. The series might be written

$$3,\ 3(2),\ 3(2)(2),\ 3(2)(2)(2),\ 3(2)(2)(2)(2)$$

or,

$$3,\ 3\cdot 2,\ 3\cdot 2^2,\ 3\cdot 2^3,\ 3\cdot 2^4$$

Let $a$ represent 3 and $r$ represent 2. Then the series could be written $a$, $ar$, $ar^2$, $ar^3$, $ar^4$.

The exponent of the quotient, or ratio, in any term is one less than the number of the term; hence, $l = ar^{n-1}$.

$$ar^{n-1} = l \quad (1) \qquad a = \frac{l}{r^{n-1}} \quad (2) \qquad r^{n-1} = \frac{l}{a} \quad (3) \qquad r = \sqrt[n-1]{\frac{l}{a}} \quad (4)$$

From the given series, $a = \frac{48}{2^4}$; $r = \sqrt[4]{\frac{48}{3}}$

The square root of the product of the first and third of three consecutive terms in G. P. will give the geometric mean; the square root of the product of the first and last of any odd number of consecutive terms in G. P. will give the geometric mean.

$$S = a + ar + ar^2 + \cdots + ar^{n-1} \quad [ar^{n-1} \text{ or } l]$$

Multiply this sum by $r$, and from the product subtract the sum.

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \quad [ar^n \text{ or } lr]$$

$$S = a \quad + ar \quad + ar^2 + ar^3 + \cdots + ar^{n-1}$$

---

$$rS - S = ar^n - a \qquad [ar^n - a \text{ or } lr - a]$$

$$(r-1)S = ar^n - a \text{ or } (r-1)S = lr - a$$

$$S = \frac{ar^n - a}{r-1} \text{ or } \qquad S = \frac{lr - a}{r-1}$$

*Example A.* Find the sum in the series 3, 6, 12, 24, 48.

$$S = \frac{3 \cdot 2^5 - 3}{2-1} \quad \text{or} \quad S = \frac{48 \cdot 2 - 3}{2-1}$$

*Example B.* The 6th term of a series is 96; the 4th term is 24. Find the ratio.

$$ar^5 = 96 \quad (1) \qquad r^2 = 4 \quad (3)$$

$$ar^3 = 24 \quad (2) \qquad r = \pm 2 \quad (4)$$

*Example C.* The 6th term of a series is 20; the 10th term is 320. Find the first term.

$$ar^9 = 320 \quad (1)$$

$$ar^5 = 20 \quad (2)$$

$$r^4 = 16 \quad (3)$$

$$r = \pm 2 \quad (4)$$

The number of terms is 10, the last term is 320. $l = ar^{n-1}$

$$a \times \pm 2^9 = 320 \quad (5)$$

$$a = \pm \tfrac{320}{512} \text{ or } \pm \tfrac{5}{8} \quad (6)$$

## EXERCISE 56

1. Find $l$. $a=2,\ r=3,\ n=5$.

2. Find $r$. $a=\frac{1}{2},\ l=8,\ n=5$.

3. Find $a$. $l=81,\ r=3,\ n=5$.

4. The first term is 7, the third is 28. Find the second term.

5. The first term is 2, the seventh is 128. Find the mean.

6. The fourth term is 8; the eighth term is 128. Find $r$, $a$, and the 9th term.

7. The first term is 1; the third term is $\frac{4}{9}$. Find the mean, the ratio, and the sum of five terms.

8. Find the sum of 2, 4, 8, ⋯ to seven terms.

9. How much could be saved in ten days by saving one cent the first day, two cents the second day, and so on?

10. Find the sum of six terms of the series 64, 32, 16, ⋯.

11. Find the seventh term of the series 5, 10, 20, ⋯.

12. Find the ratio if the first term is 2 and the third is 72.

13. What is the sixth term of the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, ⋯?

**66.** A decreasing geometric series with an unlimited number of terms is called an infinite series. The series is unending since each term after the first is a fixed part of the preceding term.

The last term approaches 0 as its limit.

If the limit 0 is substituted in the formula for $S$, for the symbol of the last term, the formula for the limit of the sum of the infinite series is obtained.

$$S = \frac{ar^n - a}{r - 1} \text{ or } \frac{a - ar^n}{1 - r}.$$

$$\textit{Limit } S = \frac{a - 0}{1 - r} \quad \text{or} \quad \frac{a}{1 - r}.$$

Given, the unending series, $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \cdots$

$$\textit{Limit } S = \frac{2}{1 - \frac{1}{3}} = 3.$$

Since 3 is obtained for the sum of the series when 2 is not diminished, 3 must be greater than a sum obtained when 2 is diminished by the last term, however small that term may be. Hence 3 is not the exact sum of the series, but it is the limit of the sum; that is, if it were possible for the last term to reach its limit the sum of the series would be 3.

**EXERCISE 57**

Each of the exercises which follow gives three terms of a decreasing geometric progression in which the ratio is less than 1 and the number of terms is infinite. The last term approaches 0.

The sum approaches what number?

1. $8, 4, 2, \cdots$
2. $8, 1, \frac{1}{8}, \cdots$
3. $1, -\frac{1}{2}, +\frac{1}{4}, \cdots$
4. $\frac{2}{9}, \frac{1}{9}, \frac{1}{18}, \cdots$
5. $6, \frac{6}{10}, \frac{6}{100}, \cdots$
6. $\frac{8}{5}, -1, \frac{5}{8}, \cdots$
7. $\frac{3}{2}, -\frac{2}{3}, \frac{8}{27}, \cdots$
8. $1, -\frac{1}{3}, \frac{1}{9}, \cdots$
9. $.3, .03, .003, \cdots$
10. $10, -6, \frac{18}{5}, \cdots$

# ANSWERS

### Exercise 1, pages 6–8

1. $0$.
2. $8x^2 + 6x + xy + 8y^2 - 8y$.
3. $10ax + 7ab + 3a^2b + 27$.
4. $-4x^3 + 7x^2 - 1$.
5. $4d - 2c + 5b + 3x - 2y$.
6. $9a^2 - c^3 + c^2 - c$.
7. $5xy + 22(m + n)$.
8. $a(m + n)^2 - 6c(x - y)^3 - n(a - b)$.
10. $16x - 6y - 3z$.
11. $16m - 11n$.
12. $3xy - 2xy^2 + x^2y^2 + 4yz - 3y^2z^2$.
13. $16ax + 3bx - 8cx$.
14. $4a^2 - 3a$.
15. $-bn - 3cp + 3cm$.
16. $a^2 + b^2$.
17. $-4(a + b)x - a(x + y) + 4by$.
18. $2a^4 + 12a^2b^2 + 2b^4$.
19. $3x^3 - 2x^2 - 9x + 3$.
20. $3b^3 + 3ab^2 + 4a^3 + 14a^2b$.
21. $x^5 + 4x^4 - 3x^3 - 20x^2 + 18$.
22. $x^5 - 5x^4y + 10x^3y^2 - 12x^2y^2 + 6xy^4 - y^5$.
23. $12a^2 - 6ab + 7b^2 + 10a - 8b$.
24. $0$.
25. $a^5 - a^3 + 5a^2 - 5a + 2$.

### § 14. Exercise 2, page 11

1. $+a^6$.
2. $+b^5$.
3. $+c^7$.
4. $-a^6$.
5. $+a^6$.
6. $-a^6$.
7. $-a^5b^4$.
8. $+3a^2bc^3$.
9. $+2a^3b^4$.
10. $+15a^2b^2$.
11. $+28ax^3y^2$.
12. $-14a^3b^4$.
13. $-24x^2y^3z$.
14. $+7x^5y^6$.
15. $-11a^5x^6$.
16. $+3a^2b^3c^3$.
17. $+a^2b^2xy$.
18. $-10a^3x^3y^2$.
19. $+72a^4b^5$.
20. $-35a^4b^4c$.

### Exercise 3, page 14

1. $b^2 - 2bd + d^2$.
2. $m^2 + 6m + 9$.
3. $c^2 + 10c + 25$.
4. $1 - 2a + a^2$.
5. $x^2 + 4x + 4$.
6. $4x^2 + 4xy + y^2$.
7. $25 - 30 + 9$.
8. $9a^2 + 12ab + 4b^2$.
9. $n^2 - 2an + a^2$.
10. $s^2 + 6s + 9$.
11. $k^2 + 2ok + o^2$.
12. $z^2 - 2xz + x^2$.
13. $a^2m^2 + 6am + 9$.
14. $a^2b^2 + 2abcd + c^2d^2$.
15. $x^2 - 2xy + y^2$.
16. $m^2 + 2mn + n^2$.
17. $c^2 - 2cz + z^2$.
18. $9 - 6x + x^2$.
19. $a^2 + 2ad + d^2$.
20. $100a^2 + 20an + n^2$.
21. $100n^2 + 20an + a^2$.

### Exercise 5, page 15

1. $x^2 + y^2 + z^2 - 2\,xy + 2\,xz - 2\,yz$.
2. $a^2 + d^2 + c^2 - 2\,ad - 2\,ac + 2\,cd$.
3. $m^2 + n^2 + c^2 + 2\,mn + 2\,mc + 2\,cn$.
4. $b^2 + l^2 + k^2 + 2\,bl + 2\,bk + 2\,lk$.
5. $p^2 + q^2 + 16 + 2\,pq - 8\,p - 8\,q$.
6. $25 + t^2 + v^2 - 10\,t - 10\,v + 2\,tv$.

### § 21. Exercise 15, page 23

1. $m^3 - n^3$.
2. $m^3 + n^3$.
3. $c^3 - 1$.
4. $c^3 + 1$.
5. $27 - x^3$.
6. $27 + x^3$.
7. $x^3 + 8$.
8. $x^3 - 8$.
9. $x^6 - y^6$.
10. $x^6 + y^6$.

### § 24. Exercise 21, page 27

1. $11\,xy^3$.
2. $5\,ab$.
3. $6\,m^2y^3$.
4. $4\,a^2c^3d$.
5. $3\,xy^3z^2$.
6. $13\,b^2c$.

### § 24. Exercise 23, pages 28, 29

1. $y^2(x - 3)$.
2. $abc - 1$.
3. $3\,x(x - 3)$.
4. $a + 1$.
5. $3(x + y)$.
6. $x - 3$.
7. $a^2(a + 7)$.
8. $9(a^2 + ab + b^2)$.
9. $9(a^4 + b^4)$.
10. $3\,c(m^2 - m^3 + 1)$.
11. $x + 2$.
12. $x + 4$.
13. $x^2 - x + 1$.
14. $5(a + b - c)$.
15. $x^2 + ax + a^2$.
16. $2\,a^2 - b$.

### § 25. Exercise 24, page 30

1. $2520\,x^3y^5z^6$.
2. $105\,c^3d^2x^5$.
3. $15(a + 2)^2(a + 3)(a + 4)$.
4. $(a^2 - 4)^2(a + 3)$.
5. $(a - 1)(a + 2)(a - 3)$.
6. $(a - 4)(a + 5)(a - 6)$.
7. $(1 - x)^4$.
8. $204(x^{12} - 1)$.
9. $105\,b^3(a^2 - b^2)$.
10. $90(x + 3)(x - 3)(x - 4)$.
11. $81(x^4 + a^2x^2 + a^4)$.

### § 26. Exercise 25, page 31

1. $-5\,m^2 - 10\,mn - 5\,n^2$.
2. $-3\,a^3 + 3\,a^2 + 3\,a$.
3. $3\,a^4b - 3\,a^3b - 3\,a^2b$.
4. $7\,a^3 - 21\,a^2b + 21\,ab^2 - 7\,b^3$.
5. $6\,a^2 - 6\,b^2$.
6. $-a^3 + 4 - m$.
7. $-11 + 33\,x + 11\,y - 11\,z$.
8. $-a^4 - c^2 - 4 + 2\,a^2c + 4\,a^2 - 4\,c$.
9. $2\,a - 3\,b + 4\,c$.
10. $2\,a - 2\,b - 2\,c$.
11. $-a^2 + b^2 + 3\,x + 3\,y$.
12. $-5\,a^2 - 10\,ab - 5\,b^2 + 1$.

### § 26. Exercise 26, pages 31, 32

1. $(a+b)^2 - 4$.
2. $\overline{a-b}^2 - 9$.
3. $(a+b)^2 - (c+d)^2$.
4. $(m+n)x + (m+n)(y+z)$.
5. $a^2 - \overline{b+c}^2$.
6. $(a+b)^4$.
7. $(x+y)x^2 - (x+y)x - (x+y)y$.
8. $16 - (2x-3)^2$.
9. $(p+q)^2 - r^2$.
10. $18a^3c^3(m^4 - n^4)$.

### § 28. Exercise 28, page 36

1. $x^2 - ax + 2a^2$.
2. $10am + 10a - 3bc - 11$.
3. $3x^3 + 4x + 16$.
4. $-5xy - 5xz + 2y^2 + yz$.
5. $7a^2 - 6ab + b^2$.
6. $2x^2 - 10x + 9$.
7. $-3x^3 + x^2y - 2xy^2$.
8. $-6a^2 + 4a - 9$.
9. $-2mn - 3xy + 13bc$.
10. $3x^2 - 11y + 3z$.

### § 29. Exercise 29, page 37

1. $1 - (8xy - 15x^3y - 13x)$.
2. $y^2 - (10xy - 15x^2y^2 + 5)$.
3. $a + 2b - (3c - 4d - m)$.
4. $x - 3y - (-2z + r - 8)$.
5. $14a + 15b - (-8c + d - m)$.
6. $-x^2 - 2y^2 - (-3z^2 - t + 11)$.
7. $3ax - (ay - 5bx - by^2)$.
8. $4a^2c^2 - (-bc - bx - b^2y)$.
9. $9xz - (yz^2 + xr^3 - yr)$.
10. $7a^3 - (a^2b - ab^2 + b^3)$.

### § 29. Exercise 30, pages 37, 38

1. $(x-y)(a+b)$.
2. $(a+b)(c+d)$.
3. $(x-y)(z-r)$.
4. $(a-b)(a^2+b^2)$.
5. $(x+y)(x^2+y^2)$.
6. $(a-y)(a^2+y^2)$.
7. $(c-b)(b-a)$.
8. $(x+y)(2x-z)$.
9. $(a+3b)(5-4x)$.
10. $(a-11)(7y-11x)$.
11. $(2c-7d)(5a+3b)$.
12. $(2a-3b)(6a+4c)$.

### § 31. Exercise 32, pages 39, 40

1. $+7a^2b^3$.
2. $-8a^2b^4c^6$.
3. $+2bc^2d^3$.
4. $-5m^2q^4$.
5. $+9a^5c^5n$.
6. $-d^2c^3n^4$.
7. $+4x^2z^2$.
8. $-6b + ac$.
9. $-3xy^2 + 2x^3z$.
10. $+11m^2 + 2c^4$.
11. $+4a^3 - 6a^2x$.
12. $-5mn^2 + 6m^2n$.
13. $+5a^2x^4y + 4ax^2z$.
14. $+2a^2by^5 + a^2b^3x$.

### § 32. Exercise 33, pages 41, 42

1. $x^2 - 2xy + y^2$.
2. $y - 1$.
3. $a^2 + ab + b^2$.
4. $x^2 + 2xy + y^2$.
5. $a^4 - a^2b^2 + b^4$.
6. $a^4 + b^4$.
7. $m^2 + 2mn + 3n^2 + \dfrac{4mn^3 - 2n^4}{m^2 - 2mn + n^2}$.
8. $9x^4 - 3x^2 + 1 - \dfrac{2}{3x^2+1}$.
9. $2 - \dfrac{8c^2 + 4c + 3}{8c^3 + 4c^2 + 2c + 1}$.
10. $x^4 + 4x^2 + 8$.
11. $x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6$.

## § 32. Exercise 34, page 42

1. $a^2 + 4$.
2. $2a - 3b$.
3. $x^2 + 5x + 6$.
4. $x^2 + 4x + 5$.
5. $a^2 - 3ab + 5b^2$.
6. $2a^2 + a - 4$.
7. $-5a^2 + 3ab - b^2$.
8. $1 - 3x - 4x^2$.
9. $1 - 8x + 7x^2$.
10. $x^2 - 3x + 10$.
11. $a^2 - ab + b^2$.
12. $a^2 - 4a + 5$.
13. $3x^2 + 2x + 4$.
14. $x^2 + 5xy + 4y^2$.
15. $x^3 - x^2 + x - 1$.
16. $27x^2 + 18y^2$. Rem., $+20y^4$.
17. $a - b$.
18. $16a^3 - 8ab^2$. Rem., $8ab^4 + b^5$.
19. $a^2 - a + 1$.
20. $a + b - c$.

## § 40. Exercise 36, pages 62, 63

1. $\frac{9a^2}{12a^3}, \frac{5}{12a^3}, \frac{8}{12a^3}$.
2. $-\frac{b(b+x)}{b^2-x^2}, \frac{bx}{b^2-x^2}, -\frac{b^2}{b^2-x^2}, \frac{x(b+x)}{b^2-x^2}$.
3. $\frac{c(c+b)(c^2+b^2)}{c^4-b^4}, \frac{bc(c^2+b^2)}{c^4-b^4}, \frac{b^2(c^2-b^2)}{c^4-b^4}, \frac{b(c-b)(c^2+b^2)}{c^4-b^4}$.
4. $\frac{5(a^2-1)}{(a^2-1)^2}, \frac{(a+1)(a^2-1)}{(a^2-1)^2}, \frac{3(a-1)(a^2-1)}{(a^2-1)^2}, \frac{4(a-1)^2}{(a^2-1)^2}, \frac{x(a+1)^2}{(a^2-1)^2}$.
5. $\frac{bx}{x^3-b^3}, \frac{x^2-b^2}{x^3-b^3}, \frac{b(x^2+bx+b^2)}{x^3-b^3}$.

## § 40. Exercise 37, page 63

1. $\frac{x+y}{2y}$.
2. $\frac{x+y}{x-y}$.
3. $\frac{x^2-ax+a^2}{x-a}$.
4. $\frac{4(x+a)}{5(x-a)}$.
5. $\frac{5a(ax-3y^2)}{2ax}$.
6. $\frac{x+2}{x+5}$.
7. $\frac{a-7}{a+3}$.
8. $\frac{c-5}{c+7}$.
9. $\frac{c+8}{c-7}$.
10. $\frac{a+12}{a+9}$.
11. $\frac{x+n}{x^2+nx+n^2}$.
12. $\frac{x^2+1}{x^4+x^2+1}$.
13. $\frac{x^3}{x^3-b^2y}$.
14. $\frac{1}{x^2-b^2}$.
15. $\frac{1}{x-a}$.
16. $\frac{3x^2y}{x-5}$.
17. $\frac{x+3}{x-6}$.
18. $\frac{4x^2-6xy+9y^2}{2x+3y}$.
19. Prime numbers.
20. $\frac{a^2+b^2}{a^2-b^2}$.

### § 40. Exercise 38, page 64

1. $\frac{3m(a+b)}{3m(a^2-b^2)}$, $\frac{15a^2cm(a+b)}{3m(a^2-b^2)}$, $\frac{12m(a+b)^2}{3m(a^2-b^2)}$, $\frac{1}{3m(a^2-b^2)}$.

2. $\frac{9bx^3y(x^2-y^2)}{b(x-y)(x+y)^3}$, $\frac{b(a+b)(x-y)}{b(x-y)(x+y)^3}$, $\frac{3c(x+y)^2}{b(x-y)(x+y)^3}$.

3. $\frac{(a^2+ab+b^2)(a+b)^2}{(a+b)^3}$, $\frac{2(a-b)(a+b)^2}{(a+b)^3}$, $\frac{5a(a^2+b^2)}{(a+b)^3}$.

4. $\frac{(m-n)(m^2-n^2)}{m^2-n^2}$, $\frac{m+n}{m^2-n^2}$, $\frac{3(a+b)(m-n)}{m^2-n^2}$.

5. $\frac{15x}{15x(a^4+b^4)}$, $\frac{3}{15x(a^4+b^4)}$, $\frac{5ab^2c(a^4+b^4)}{15x(a^4+b^4)}$.

6. $\frac{a+3}{(a+2)(a+3)}$, $\frac{a+3}{(a+2)(a+3)}$, $\frac{a+2}{(a+2)(a+3)}$.

7. $\frac{a(a^3-b^3)}{b(a^3-b^3)}$, $\frac{a^3b(a^2+ab+b^2)}{b(a^3-b^3)}$, $\frac{b^3(a-b)}{b(a^3-b^3)}$.

8. $\frac{15a(c-d)(d-a)}{15a(a-b)(b-c)(c-d)(d-a)}$, $\frac{15a(a-b)(b-c)}{15a(a-b)(b-c)(c-d)(d-a)}$, $\frac{(a-b)(b-c)(c-d)(d-a)}{15a(a-b)(b-c)(c-d)(d-a)}$.

9. $\frac{18}{2(16-b^2)}$, $\frac{16-b^2}{2(16-b^2)}$, $\frac{2(4+b)}{2(16-b^2)}$.

10. $\frac{a-3x}{a^2-5ax+6x^2}$, $\frac{1}{a^2-5ax+6x^2}$, $\frac{a-2x}{a^2-5ax+6x^2}$.

### § 41. Exercise 39, page 67

1. $\frac{4x-6y}{5}$.

2. $\frac{3d-c}{12}$.

3. $\frac{4a-1}{1-a^2}$.

4. $\frac{n^2-2n}{n^2-9}$.

5. $\frac{10x-a}{6}$.

6. $\frac{32a+31}{36}$.

7. $\frac{13a}{a^2-x^2}$.

8. $\frac{1}{x+5}$.

9. $\frac{2(a^2+1)}{a^2-1}$.

10. $\frac{a+b+2}{(a+b)^2}$.

11. $\frac{n^2(x+z)-mn(x-z)}{x^2-z^2}$.

12. $\frac{3c^2+11x^2}{x^4-c^4}$.

13. $\frac{11x^3-18x^2-27x-16}{30x^3}$.

14. $\frac{d^2+3x^2}{2ax}$.

15. $\frac{8ab}{a^2-4b^2}$.

16. $\frac{x^2-y^2}{xy^3}$.

17. $\frac{1+x^2-3x^3-x^5}{1-x^6}$.

18. $\frac{2+a^2+ab+b^2+3(a-b)}{a^3-b^3}$.

19. $\frac{4-x^2}{x^4-1}$.

20. $\frac{a+x-1-3(x^2-ax+a^2)}{a^3+x^3}$.

21. $\frac{3b+2}{(b-2)(b^2-1)}$.

22. 0.

23. 1.

24. $\frac{2(a+b)}{(a-c)(b-c)}$.

25. $-\frac{x+z}{y+z}$.

26. $\frac{x-6}{(x-1)(x-3)}$.

27. $\frac{1}{acx}$.

28. $\frac{(x^4-y^4)+(x^4+y^4)(x^2-y^2)+2x(x^4+y^4)(x^2+y^2)}{x^8-y^8}$.

## § 42. Exercise 40, page 70

1. $\frac{5a^5+32a+12}{5}$.

2. $\frac{x^2+x-xy-3y-y^2}{x-3y}$.

3. $\frac{x^2+xy+y^2}{x+a}$.

4. $\frac{1}{1-x}$.

5. $\frac{7a^3-2}{2a-1}$.

6. $\frac{2x^4+3x^2+2}{x^2+x+1}$.

7. $\frac{2x^2-5xy+13y^2}{3x-2y}$.

8. $\frac{2a^2-10ax-3x^2}{4a+3x}$.

9. $\frac{25x-11xy-15y+4y^2}{5x-3y}$.

10. $\frac{2a^3+3a}{a^2-a+1}$.

11. $\frac{a^8-3a^4-3a^2-4}{a^4+1}$.

12. $\frac{11x-5a}{4}$.

13. $\frac{a^3}{a+x}$.

14. $\frac{4a^3-2a^2+2}{3}$.

15. $\frac{15xy-n^3+n^2+n+1}{3xy}$.

16. $\frac{8m^2(1-m)-(a-b)^2}{1-m}$.

17. $\frac{8a^2+3a-m^4-2am^2}{3a}$.

18. $\frac{(a+b)^3-1}{a+b}$.

## § 43. Exercise 41, page 71

1. $\frac{ad}{1+a}$.

2. $\frac{37(a-1)}{a+1}$.

3. $am(a-b)$.

4. $\frac{a-b}{a^3+b^3}$.

5. $\frac{(x^4+a^4)(x^2+xa+a^2)}{5(x^2+a^2)}$.

6. $\frac{m+n}{7}$.

7. $3(x^2-a^2)$.

8. $\frac{3(a+b)}{a-b}$.

9. $5a^2b^2$.

10. $\frac{a+b-c-d}{a^3-b^3}$.

11. $3ab(a-b)$.

12. 3.

13. $\frac{am-3}{a^2m^2-6am}$.

14. $\frac{a+2}{a-2}$.

15. $\frac{9+3x+x^2}{33}$.

16. $\frac{3a(x+2)}{(x+7)(x-2)}$.

17. $\frac{1}{a-6}$.

18. $\frac{25(a+b-c)}{3}$.

19. $\frac{abc+1}{abc-3}$.

**20.** $121\,a^2b^2c^2$. **21.** $abc(b-a)$. **22.** $\frac{1-9\,a^2}{1-a^2}$. **23.** $x$.

**24.** $\frac{2\,a+b}{b^2}$. **25.** $\frac{a(a^2+ab+b^2)}{a+b}$. **26.** $a^2-b^2$.

**27.** $\frac{b^2(a+b)}{2\,a^2}$. **28.** $28\,x+35\,c+24$. **29.** $-\frac{ax}{mn}$.

**30.** $-\frac{5\,b^2}{a-b}$. **31.** $\frac{50(2\,x^2-7\,x+3)}{7(x+1)}$. **32.** $\frac{2}{y}$.

### § 44. Exercise 42, page 74

**1.** $\frac{a-2}{a-4}$. **5.** $\frac{ax}{2\,a-1}$. **9.** $\frac{a+3}{a-3}$.

**2.** $\frac{x^2+nx+n^2}{x^2-nx+n^2}$. **6.** $\frac{a(a+4)}{a+1}$. **10.** $\frac{c-2}{c+1}$.

**3.** $\frac{7\,a-1}{2(2\,a-1)}$. **7.** $\frac{1}{a^4-x^4}$. **11.** $\frac{x-1}{a+2}$.

**4.** $\frac{4\,a+3\,b}{a+2}$. **8.** $\frac{ax(x-1)}{x-7}$. **12.** $-\frac{c+a}{3(c^2+c+1)}$.

**13.** $\frac{x+b}{x^2+b^2}$. **15.** $\frac{n-o}{211}$.

**14.** $\frac{(a+3)(a^2-a+1)}{(a-3)(a^2+a+1)}$. **16.** $\frac{a^2-2\,a+4}{a-2}$.

### § 45. Exercise 43, page 75

**1.** $\frac{acx+bc}{acx-bx}$. **3.** $\frac{1}{b-a}$. **5.** $\frac{a-1}{a-7}$.

**2.** $\frac{x-1}{x+3}$. **4.** $\frac{a^2c-ab^2-bc^2}{ab-bc-ac}$. **6.** $\frac{a+b}{a-b}$.

**7.** $\frac{1}{x}$. **10.** $c-1$. **13.** $\frac{9\,a^2-a^4}{2\,a-1}$.

**8.** $\frac{x^2(x-3)}{2-x}$. **11.** $\frac{a^3+ab^2}{b^4-ab^2}$. **14.** $\frac{b^3+c^3}{b^2+c^2}$.

**9.** $\frac{3\,a-n}{2\,x-3\,n}$. **12.** $\frac{9\,x+60}{15\,z^2+10\,xz}$. **15.** $\frac{mn^3+m^3n}{n^2-m^2}$.

**16.** $(a-b)(a^2+c^2)$.

### § 46. Exercise 44, page 78

**8.** $\frac{(a^2-b^2)^2}{9\,c^2(a^2+b^2)^2}$. **9.** $\frac{(a^2+ac+c^2)^2}{(a+c)^2}$. **10.** $(a^4+b^4)^3$.

**Exercise 45, pages 79, 80**

**8.** $11 - \frac{x}{c}$.

**9.** $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{2}{ab} + \frac{2}{ac} + \frac{2}{bc}$.

**10.** $\frac{1}{x^3} - \frac{3}{x^2y} + \frac{3}{xy^2} - \frac{1}{y^3}$.

**11.** $\frac{1}{a^4} - \frac{4}{a^3b} + \frac{6}{a^2b^2} - \frac{4}{ab^3} + \frac{1}{b^4}$.

**15.** $8a^3 + 84a^2 + 294a + 343$.

**16.** $8a^3c^3 - 12a^2c^2x + 6acx^2 - x^3$.

**17.** $a^6x^3 - 6a^4cx^2 + 12a^2c^2x - 8c^3$.

**18.** $16x^4 - 32a^2x^3 + 24a^4x^2$
$- 8a^6x + a^8$.

**19.** $a^8 - 12a^6b + 54a^4b^2$
$- 108a^2b^3 + 81b^4$.

**20.** $1 + 8x + 24x^2 + 32x^3 + 16x^4$.

**21.** $x^8 - 4x^6y^2 + 6x^4y^4 - 4x^2y^6 + y^8$.

**22.** $\frac{8x^3}{27} - x^2y + \frac{9xy^2}{8} - \frac{27y^3}{64}$.

**23.** $\frac{x^3}{8} + \frac{3x^2}{4} + \frac{3x}{2} + 1$.

**24.** $\frac{a^4}{81} - \frac{2a^3x}{27} + \frac{a^2x^2}{6} - \frac{ax^3}{6} + \frac{x^4}{16}$.

**§ 47. Miscellaneous Exercises, pages 82–88**

**4.** $\frac{x^2}{x^2 - z^2}$; $\frac{y^2}{(y-x)^2}$; $\frac{z^2}{(z-x)(z-y)}$.

**6.** $x^{12} - y^{12}$.

**10.** $3 - \frac{7x^2 - 5x - 2}{2x^3 + x^2 - x - 2}$.

**11.** $\frac{2a^4 + 3a^2 + 2}{a^2 + a + 1}$.

**12.** $\frac{3n + 2}{(n-2)(n^2-1)}$.

**13.** $\frac{8x^7}{a^8 - x^8}$.

**16.** $2x - 6x^3 + 18x^5 - 54x^7 + \frac{162x^9}{1 + 3x^2}$.

**18.** $\frac{1}{2y} - \frac{1}{x} + \frac{3y}{2x^2}$.

**19.** $5c - 4d - (5b + 3c - 2d)$.

**20.** $\frac{3b^2}{3a^2 - 2ab}$.

**22.** $-15 - 11m$.

**23.** $99x^3 + 179x^2 + 53$ cents.

**24.** $5x - 3a = 5x - 3a$.

**25.** $3a^2c$ dollars.

**33.** $2r^2 + 2 - \frac{2}{r+1}$.

**34.** $-\frac{m^2x}{(a-b)^2}$.

**35.** $1 + x + x^2 + x^3 + \cdots$.

**36.** $m^{12} - n^{12} + 2n^4$.

**37.** (1) $x^2y^2 + y^2z^2 + x^2z^2 + 2xy^2z + 2x^2yz + 2xyz^2$;
(2) $x^2 \pm 2xy + y^2$; (3) $a^2 \pm 10a + 25$.

**38.** $\frac{x+y}{x-y}$ or $\frac{x-y}{x+y}$.

**40.** 0.

**41.** $\frac{a^2 + b^2}{a}$.

**48.** $[(a^2 + bx) - (ax + ab)][(a^2 + bx) + (ax + ab)]$.

**50.** $-4xy - 4xz$.

**52.** $c = 7$.

**65.** $210\,M^6R^4$. **66.** $\dfrac{c+d}{2x+f}$. **67.** $\dfrac{a-n}{m+1}$, $\dfrac{am+n}{m+1}$.

**68.** $\dfrac{(96)^3}{x^3} - x^3 : \left(\dfrac{96}{x} - x\right)^3 = 19 : 1$. **78.** $\dfrac{m+n}{2}$, $\dfrac{m-n}{2}$.

## § 48. Problems, pages 89–94

**1.** 5 plants.
**2.** 18 volumes.
**3.** 7 canoes.
**4.** 100 lanterns.
**5.** 6 passengers.
**6.** 1737, the date.
**7.** 8 years.
**8.** 1671, the date.
**9.** \$120.
**10.** 18 boys.
**11.** 33, 55, 36, fish.
**12.** 80, 81, numbers.
**13.** 6 given away.
**14.** \$700, \$1400, \$2800, shares.
**15.** $3\frac{3}{4}$ miles an hour.
**16.** 4 oranges, $\frac{1}{2}$ cent.
**17.** 13, 7; $\dfrac{m+n}{2}$, $\dfrac{m-n}{2}$.
**18.** 10 men, 15 men; $\dfrac{qr}{p-q}$, $\dfrac{pr}{p-q}$.
**19.** 17 girls.
**20.** 98 windows.
**21.** 15 papers.
**22.** $1\frac{3}{11}$ minutes.
**23.** 6 days.
**24.** \$800, \$1000, spent.
**25.** 4 were poor.
**26.** 28 maps.
**27.** 65 cents.
**28.** $1\frac{1}{6}$, $8\frac{1}{6}$, numbers.
**29.** $13\frac{5}{9}$ units.
**30.** $\dfrac{bc}{b+c}$ hours.
**31.** 50 sheep.
**32.** $\frac{13}{17}$, the fraction.
**33.** $\dfrac{np-q}{m(np-q)}$, the fraction.
**34.** 186 gallons.
**35.** 3 feet, 14 feet.
**36.** $\dfrac{mnr}{m-n}$ ft., $\dfrac{2\,mnr}{33(m-n)}$ rods.
**37.** 16 ft., 19 ft.
**38.** 8 half-dollars, 10 quarter-dollars.

## § 51. Problems Leading to Simultaneous Equations, pages 100–103

**1.** 55 years, 21 years, the ages.
**2.** $\frac{28}{45}$, the fraction.
**3.** $\frac{7}{11}$, the fraction.
**4.** 42, the number.
**5.** 12, the number.
**6.** 56, 37, the numbers.
**7.** 7 hours, 8 hours.
**8.** $1\frac{1}{2}$ cents, 2 cents.
**9.** 9, 18, exercises.
**10.** 106 trout, 2 perch.
**11.** 25 cents, 50 cents.
**12.** $\frac{7}{8}$, the fraction.
**13.** $3\frac{3}{13}$ days.
**14.** A, 6 days; B, 12 days.
**15.** A, 42 days; B, 28 days.
**16.** C, 6 days; B, 4 days.

**17.** 70 yards, 50 yards.
**18.** 12, 14, the numbers.
**19.** Each, 6 days.
**20.** C, $\dfrac{pv}{v-p+q}$; D, $\dfrac{pv}{p-q}$.
**21.** 24 and 36.
**22.** 21, 23, ⋯, 39.
**23.** $5 and $4.
**24.** 44 and 33.
**25.** 10, 22, 14, the number of notes.

### § 52. Page 107

**1.** 757, the number.
**2.** A, 5; B, 4; C, 6.
**3.** $33\frac{1}{3}$ cents; 75 cents; $1.

### § 55. Exercise 49, page 119

**1.** 5, − 1
**2.** 6, 4
**3.** 5, $-\frac{11}{3}$
**4.** 5, 1
**5.** 5, − 3
**6.** 10, − 29
**7.** 4, 0
**8.** $11\frac{2}{3}$, 0
**9.** 2, $-5\frac{1}{3}$
**10.** $\frac{4}{3}$, − 3
**11.** 8, − 1
**12.** 2, − 16
**13.** 5, $-1\frac{1}{6}$
**14.** 55, − 30

### § 55. Exercise 50, pages 119, 120

**1.** 9, − 8
**2.** 20, − 11
**3.** 6, $-\frac{16}{3}$
**4.** $\frac{1}{3}$, − 4
**5.** 3, $-\frac{7}{5}$
**6.** $\frac{5}{3}$, − 3
**7.** $\frac{3}{2}$, − 5
**8.** 1, $-\frac{7}{8}$
**9.** 11, 0
**10.** 2, 1
**11.** 4, − 2
**12.** 3, 2
**13.** 3, − 2
**14.** 5, − 1
**15.** $-2a$, $-3a$
**16.** $2p$, $-3p$
**17.** $a \pm b$

### § 56. Problems Leading to Quadratic Equations, pages 120–122

**1.** 16 barrels.
**2.** 30 min., 20 min.
**3.** 32 senators.
**4.** 8 from Penn.
**5.** 7 from Va.
**6.** 252 thousands.
**7.** 12 letters.
**8.** 15 millions.
**9.** $\frac{1}{6}$ of a dollar.
**10.** 25 stars.
**11.** 5 miles an hour.
**12.** 35, − 15, the numbers.
**13.** 543, the number.
**14.** $\frac{3}{8}$, the fraction.
**15.** 24 furlongs an hour.
**16.** $600, the salary.
**17.** 1822, the date.
**18.** 14 colleges.
**19.** 195 pounds.

### § 57. Exercise 51, page 123

**1.** $x = -8, 3.$
**2.** $x = 7, -2.$
**3.** $x = 9, -3.$
**4.** $x = -5, -2.$
**5.** $x = 5, 4.$
**6.** $x = 2, 1.$
**7.** $x = 5, -1.$
**8.** $x = 4, 3.$
**9.** $x = 5, 3.$
**10.** $x = 11, -1.$
**11.** $x = 5, -2.$
**12.** $x = \frac{3}{2}, -\frac{3}{2}.$
**13.** $x = 9, 5.$
**14.** $x = -8, -3.$
**15.** $x = -\frac{2}{5}, \frac{3}{5}.$
**16.** $x = \pm 3, \pm 2.$

## § 59. Exercise 53, page 129

1. $b - 1$.
2. $n - 1$.
3. $2b - 1$.
4. $m - 2$.
5. $3ab(b - 1)$.
6. $n - 2$.

## § 62. Exercise 54, page 132

1. $a$.
2. $\frac{4}{3(b+1)}$.
3. $\frac{a^3 - a^2 + a + 1}{2a^3 - a^2 + 1}$.
4. $\frac{a^2 - 2a + 1}{2a - 1}$.
5. $\frac{bd + dnx}{an + cd + cnx}$.
6. $1 - n$.
7. $\frac{a^3 - 6a^2 + 10a - 3}{a^3 - 7a^2 + 12a - 3}$.
8. $\frac{(a^3 + b^3) - (a - b)}{a^2 - ab + b^2}$.
9. $\frac{n^4 + n^2 + n}{n^2 + 1}$.
10. $\frac{3bcd - bc - bd - cd}{bc + bd + cd}$.
11. $\frac{(x + y + z)^2}{2yz}$.
12. $\frac{a^2 + x^2}{(a + x)^2}$.

## § 64. Exercise 55, page 135

1. $a = 18$; $d = 6$.
2. $a = 19$; $d = 3$.
3. $l = 20$.
4. $a = 6$.
5. $d = 3$.
6. $n = 6$.
7. $s = 779$.
8. $l = 30$.
9. $a = 7$.
10. $s = 80$.
11. $s = 51$.

## § 65. Exercise 56, page 138

1. $l = 162$.
2. $r = \pm 2$.
3. $a = 1$.
4. $ar = \pm 14$.
5. $ar^3 = \pm 16$.
6. 256.
7. $s = \frac{211}{81}$, or $-\frac{55}{81}$.
8. $s = 254$.
9. $s = 1023$.
10. $s = 126$.
11. $ar^6 = 320$.
12. $r = \pm 6$.
13. $ar^5 = \frac{16}{243}$.

## § 66. Exercise 57, page 139

1. 16.
2. $9\frac{1}{7}$.
3. $\frac{2}{3}$.
4. $\frac{4}{9}$.
5. $6\frac{2}{3}$.
6. $\frac{64}{65}$.
7. $\frac{27}{26}$.
8. $\frac{3}{4}$.
9. $\frac{1}{3}$.
10. $6\frac{1}{4}$.